Solving the Inverse Reliability Problem Using Decomposition Techniques

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Abstract

The paper introduces a new general method for solving the inverse reliability problem with multiple design variables and constraints that are specially suitable for practitioner engineers. We seek to determine the unknown parameters such that prescribed first-order reliability indices are obtained. The method uses standard optimization frameworks to obtain the reliability indices and a decomposition iterative scheme to solve the global problem. The sensitivities with respect to the parameters are also obtained by means of the dual variables, so that the method not only attains the solution parameters of the problem, but also shows how sensitive they are with respect to the reliability indices. In addition, the proposed algorithm detects the no-solution case and the existence of infinite solutions. In this case, it also gives the local increments of each design variable to obtain new solutions. Next, the inverse reliability problem is extended to include reliability bounds and other constraints. The proposed methods are illustrated by applications to several cases described in the literature and to the problem of bridge crane design.

Key Words: Bridge crane, Generalized inverse reliability problem; Level II (FORM) methods; Optimization; Reliability-based design; Sensitivity analysis; Structural reliability.

1 Introduction and Motivation

Engineering design of structural elements is a complicated and highly iterative process that usually requires an extensive experience. Iterations consist of a trial-and-error selection of the design variables or parameters, together with a check of the safety and functionality constraints, until reasonable structures, in terms of cost and safety, are obtained.

Safety of engineering structures is a fundamental criterion for design (see Blockley [1], Ditlevsen and Madsen [2], ROM [3], Freudenthal [4], Madsen, Krenk and Lind [5], Melchers [6], Stewart and Melchers [7], Wirsching and Wu [8], Wu, Burnside and Cruse [9]). To this end, and using a probability based approach, engineers first identify all failure modes and statistical parameters (e.g., the means and standard deviations of the random variables involved) of the structure being designed, and then specify the safety constraints that have to be satisfied.

In this paper we assume that the reader is familiar with level II methods (FORM) ([4, 10, 11, 12, 13]) for evaluating the reliability index associated with any mode of failure:

$$\beta = \underset{\boldsymbol{z}}{\text{Minimum}} \quad \sqrt{\sum_{r=1}^{s} z_r^2} \tag{1}$$

subject to

$$\begin{aligned} g(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) &= 0, && (2)\\ \mathbf{T}(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\kappa}) &= \boldsymbol{z}, && (3)\\ \mathbf{q}(\boldsymbol{x}, \boldsymbol{\eta}) &= \boldsymbol{\psi}, && (4) \end{aligned}$$

where $g(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) = 0$ is the failure condition, $\mathbf{T}(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\kappa})$ is the transformation (Rosenblatt [14], Nataf [15]) giving the values of the standard and independent normal variables $\boldsymbol{z}$ as a function of the values of the random variables $\boldsymbol{x}$, the random parameters (means, characteristic or experimental values) of the design variables $\boldsymbol{\eta}$, and the variability parameters of other variables $\boldsymbol{\kappa}$ of their joint probability distribution of the random variables, $\mathbf{q}(\boldsymbol{x}, \boldsymbol{\eta}) = \boldsymbol{\psi}$ are the equations that allow obtaining the values of the intermediate variables $\boldsymbol{\psi}$ (they simplify the statement of the problem) and s is the number of random variables involved in the problem.

For example, $\boldsymbol{\eta}$ may be the set of means or characteristic values that are fixed by the engineer or by the code (e.g., material properties such as unit weights, strength, Young modula, etc.), geometric dimensions of the work being designed, or might represent unknown thresholds in limit-state functions. The parameters $\boldsymbol{\kappa}$ may be the set of parameters defining the random variability and dependence structure of the variables involved (e.g., standard deviations, correlation coefficients, etc.).

The probability of failure p_f is related to the reliability indices by the approximate relation $p_f = \Phi(-\beta)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable.

We point out the reader that the problem in (1)–(4) can give the wrong answer, that is, a negative value of β when the correct answer is a positive β. This is due to the fact that two square roots are possible in (1). A method to avoid this problem is explained in Section 2 (see the problem defined by (10)–(15)).

Definition 1 (Inverse reliability problem.) *The inverse reliability problem consists of finding the values of given subsets of $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ such that the associated reliability indices $\beta_i; i = 1, 2, \ldots, n$ (the target values) attain desired reliability levels $\beta_i^*; i = 1, 2, \ldots, n$.*

From Definition 1, the inverse reliability problem involves finding some/all of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ that give the desired reliability levels $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_n^*)^T$, where n is the number of failure modes, and

$$\beta_i^* \leq \underset{\boldsymbol{z}_i}{\text{Minimum}} \quad \sqrt{\sum_{r=1}^{s} z_{ir}^2}\ , \quad i = 1, \ldots, n, \tag{5}$$

subject to

$$\begin{aligned} g_i(\boldsymbol{x}_i, \boldsymbol{\eta}, \boldsymbol{\psi}_i) &= 0, & (6)\\ \mathbf{T}(\boldsymbol{x}_i, \boldsymbol{\eta}, \boldsymbol{\kappa}) &= \boldsymbol{z}_i, & (7)\\ \mathbf{q}(\boldsymbol{x}_i, \boldsymbol{\eta}) &= \boldsymbol{\psi}_i, & (8) \end{aligned}$$

where the constraint in (6) is the limit state equation corresponding to the failure mode i.

Additional constraints corresponding to geometric or code constraints, lower and upper bounds, fixed values, etc., can be included

$$h_j(\boldsymbol{\eta}, \boldsymbol{\kappa}) \leq 0; \; j = 1, 2, \ldots, m. \qquad (9)$$

The inverse reliability problem stated above is more general than the inverse reliability problem normally stated in the existing literature, in the sense that (a) it includes constraints as those in (9), and (b) there is an inequality ($\leq$) in (5) to fix minimum safety requirements instead of the usual equality.

The standard and simpler reliability problem has strong constraints that can not always be satisfied, even though the number of design variables is smaller or equal to the number of constraints in the problem. However, the engineer is not interested in having fixed reliability indices, but reliability indices above some given lower bounds. This, apart from leading to a more reasonable engineering problem, releases the problem from unnecessary mathematical difficulties, and makes the existence of solutions possible in many cases. The complexity of solving the inverse reliability problem, however, is due to the constraints in (5) that are themselves optimization problems.

The inverse reliability problem for one parameter can be solved by "trial and error"; solving repeatedly (1)–(4) and interpolating the design parameters at the desired reliability. This method is obviously tedious and more efficient solutions have been proposed. For example, Winterstein, Ude, and Cornell [16] describe a reliability contour method applied to problems in offshore environmental loads in the context of limit-state functions of the form $g(\boldsymbol{x}, \theta) = \theta - h(\boldsymbol{x})$, where θ is a given threshold treated as a deterministic design variable.

To extend the method to general limit states, Der Kiureghian, Zhang and Li [17] propose an extension algorithm of the well-known Hasofer-Lind-Rackwitz-Fiessler algorithm used in reliability analysis, where the search space of the random variables is extended to include the unknown parameters. A search direction and a penalty function are introduced to guide the sequence of iterations to the design point and the parameter solution in a balanced manner. Li and Foschi [18, 19] present a direct algorithm for the single-parameter inverse problem and use a Newton-Raphson iterative algorithm for finding multiple design parameters. Thus, this general method solves inverse reliability problems with multiple design points and constraints. The design parameters can also be treated as random variables, so that in these cases they could be the means and/or the standard deviations of the design variables (the $\boldsymbol{\eta}$ variables). Alternatively, Sadovský [20] proposes some modifications of the algorithm presented by Li and Foschi [18, 19] that may improve convergence of the design parameter, and suggests an alternative algorithm including curvature information that shows a more robust behavior, but it may require additional computational effort.

The solutions proposed by these authors are valid and have good properties but they do have some shortcomings for practitioner engineers. Examples of these difficulties are:

1. The user has to be acquainted with optimization algorithms in-depth, and it could be difficult to include additional bounds for variables or constraints.

2. In some cases, to achieve convergence, the selection of the constant step size is crucial. Thus, these algorithms may fail to converge if the step size is not assessed properly.

3. Sometimes sophisticated line searches complicate the method, though improve its performance.

4. No sensitivities of the reliability coefficients with respect to the design variables are given.

5. No information is given in cases where either there is no solution or there are multiple solutions.

In this paper we propose a method that uses standard optimization frameworks to obtain the reliability indices and a very simple decomposition iterative scheme to solve the global problem, that by means of the dual variables also gives the sensitivities, i.e., the partial derivatives of the β-values with respect to the parameters. In addition it detects the existence of a unique, multiple or no-solution cases. In this way, practitioner engineers can use the widely available, powerful optimization algorithms that avoid the selection of step sizes by the designer. Moreover, in practice the size of a design problem can be very large and one can encounter problems with a huge number of equations and/or unknowns. In the proposed method we utilize some special decomposition techniques to solve these problems.

The remainder of this paper is structured as follows. Section 2 presents a new method for solving the inverse reliability problem. Section 3 demonstrates the validity and efficiency of the proposed method by some numerical examples. Section 4 provides some extensions of the proposed method, including optimization. Section 5 gives an application to an engineering design problem, namely, a bridge crane design. Finally, some conclusions are drawn in Section 6.

2 The Proposed Method

We start with the simpler inverse reliability problem (with equalities in (5) and without (9)), i.e., finding the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ that give the desired reliability levels $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_n^*)^T$, where n is the number of failure modes, and

$$\beta_i^* = \underset{\boldsymbol{z}_i}{\text{Minimum}} \quad \sqrt{\sum_{r=1}^{s} z_{ir}^2} \;, \quad i = 1, \ldots, n, \tag{10}$$

subject to

$$\begin{aligned} g_i(\boldsymbol{x}_i, \boldsymbol{\eta}, \boldsymbol{\psi}_i) &= 0, && (11)\\ \mathbf{T}(\boldsymbol{x}_i, \boldsymbol{\eta}, \boldsymbol{\kappa}) &= \boldsymbol{z}_i, && (12)\\ \mathbf{q}(\boldsymbol{x}_i, \boldsymbol{\eta}) &= \boldsymbol{\psi}_i, && (13) \end{aligned}$$

$$
\begin{aligned}
g_i(\boldsymbol{x}_1, \boldsymbol{\eta}, \boldsymbol{\psi}_1)\, u_i &> 1, &(14)\\
T(\boldsymbol{x}_1, \boldsymbol{\eta}, \boldsymbol{\kappa}) &= \mathbf{0}. &(15)\\
\mathbf{q}(\boldsymbol{x}_1, \boldsymbol{\eta}) &= \boldsymbol{\psi}_1, &(16)
\end{aligned}
$$

where the auxiliary variable u_i and the three constraints in (14)-(16) ensure that the sign of β is the desired one and $\boldsymbol{x}_1$ is the random variable values corresponding to the point $\boldsymbol{z}_1 = \mathbf{0}$ in the standard normal random space.

The positive β_i^* value resulting from the solution of problem (10)–(16) must be corrected by the sign of u_i. This corrects the possibly wrong answer obtained by solving the problem (1)–(4), as mentioned earlier.

The problem in (10)-(16) is difficult to solve in the sense that it cannot be implemented directly using standard optimization packages. To solve this problem, we propose a decomposition technique. Decomposition techniques (see Mínguez [21] and Conejo et al. [22]) consists of dividing the initial problem in several problems that are easy to solve and such that iteration of solutions of these simpler problems lead to the solution of the initial problem . Thus, the proposed method consists of an iterative procedure that solves two types of problems until convergence: *the master problem*, which consists of solving a linear system of equations, and *the subproblems*, which are the evaluations of the reliability indices for all failure modes.

2.1 The Master Problem

Specifically, for iteration j, we write:

$$
\beta_i^{(j)} = f_i(\boldsymbol{\eta}^{(j)}, \boldsymbol{\kappa}^{(j)}); \quad i = 1, \ldots, n, \tag{17}
$$

where $f_i(\boldsymbol{\eta}, \boldsymbol{\kappa})$ is the function that gives the β_i-value as a function of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$. Expanding the function (17) in Taylor series, we obtain the linear system of n equations:

$$
\beta_i^* = \beta_i^{(j)} + \sum_{\forall m} \frac{\partial f_i(\boldsymbol{\eta}, \boldsymbol{\kappa})}{\partial \eta_m} \left(\eta_m - \eta_m^{(j)}\right) + \sum_{\forall l} \frac{\partial f_i(\boldsymbol{\eta}, \boldsymbol{\kappa})}{\partial \kappa_l} \left(\kappa_l - \kappa_l^{(j)}\right), \tag{18}
$$

$i = 1, 2, \ldots, n$. Thus, for iteration j the master problem consists of solving the linear system of equations (18), i.e., we find the values of η_m and κ_l that satisfy (18).

2.2 The Subproblems

In the j-th iteration, the subproblems consist of determining the reliability coefficients, i.e., solving the corresponding minimization problems. However, since for stating this linear system of equations in the master problem, we need the partial derivatives of the function (17) with respect to the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ evaluated at $\boldsymbol{\eta}^{(j)}, \boldsymbol{\kappa}^{(j)}$, we slightly modify the subproblems to obtain these partial derivatives, which are the sensitivities of the β values to $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$. To this end, we need to convert the data values into artificial variables $(\boldsymbol{\eta}_0, \boldsymbol{\kappa}_0)$

and add the corresponding constraints to the problem in (10)–(16), leading to the modified subproblems:

$$\beta_i^{(j)} = \underset{\boldsymbol{z}_i, \boldsymbol{\eta}_0, \boldsymbol{\kappa}_0}{\text{Minimum}} \quad \sqrt{\sum_{r=1}^{s} z_{ir}^2}\ , \quad i = 1, \dots, n, \tag{19}$$

subject to

$$\begin{aligned}
g_i(\boldsymbol{x}_i, \boldsymbol{\eta}_0, \boldsymbol{\psi}_i) &= 0, & (20)\\
\mathbf{T}(\boldsymbol{x}_i, \boldsymbol{\eta}_0, \boldsymbol{\kappa}_0) &= \boldsymbol{z}_i, & (21)\\
\mathbf{q}(\boldsymbol{x}_i, \boldsymbol{\eta}_0) &= \boldsymbol{\psi}_i, & (22)\\
g_i(\boldsymbol{x}_1, \boldsymbol{\eta}_0, \boldsymbol{\psi}_1)\ u_i &> 1, & (23)\\
T(\boldsymbol{x}_1, \boldsymbol{\eta}_0, \boldsymbol{\kappa}_0) &= \mathbf{0}, & (24)\\
\mathbf{q}(\boldsymbol{x}_1, \boldsymbol{\eta}_0) &= \boldsymbol{\psi}_1, & (25)\\
\boldsymbol{\eta}_0 &= \boldsymbol{\eta}^{(j)}, \quad \text{(the corresponding dual variable is } \boldsymbol{\lambda}^{(j)}\text{)}, & (26)\\
\boldsymbol{\kappa}_0 &= \boldsymbol{\kappa}^{(j)}, \quad \text{(the corresponding dual variable is } \boldsymbol{\theta}^{(j)}\text{)}. & (27)
\end{aligned}$$

Thus one can obtain the desired partial derivatives, as the values of the dual variables associated with the constraints in (26) and (27). They are the sensitivities of the objective function with respect to the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$, i. e., they give how much the objective function changes with a very small increment of the corresponding parameter. These values are nothing more than the values of the corresponding values of the dual variables.

2.3 The Modified Master Problem

Let $\boldsymbol{\alpha}$ be a vector whose i-th element is $\alpha_i = \text{sign}(u_i)$. The master problem in (18) can be transforms to:

$$\beta_i^* = \alpha_i \beta_i^{(j)} + \sum_{\forall m} \alpha_i \lambda_m^{(j)} \left(\eta_m - \eta_m^{(j)}\right) + \sum_{\forall l} \alpha_i \theta_l^{(j)} \left(\kappa_l - \kappa_l^{(j)}\right); \quad i = 1, \dots, n, \tag{28}$$

where the sign function is used to correct for the sign of β and allows the updating of the design parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ until convergence is reached. This process is illustrated in Figure 1 for the simple case of a single parameter η.

Note that the system of n equations (28) in $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ can be written in matrix form as:

$$\mathbf{A} \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\kappa} \end{pmatrix} = \boldsymbol{\beta}^* - \boldsymbol{\beta}^{(j)} + \mathbf{A} \begin{pmatrix} \boldsymbol{\eta}^{(j)} \\ \boldsymbol{\kappa}^{(j)} \end{pmatrix} = \boldsymbol{\gamma}. \tag{29}$$

This system can have no solution, a unique solution or multiple solutions. Therefore, it is important to know when each of these situations occurs:

1. No solution: This occurs when $\text{rank}(\mathbf{A}) \neq \text{rank}\,(\mathbf{A}|\boldsymbol{\gamma})$, where $\mathbf{A}|\boldsymbol{\gamma}$ is the matrix $\mathbf{A}$ augmented by the vector $\boldsymbol{\gamma}$.

2. An unique solution: When $\text{rank}(\mathbf{A}) = \text{rank}\,(\mathbf{A}|\boldsymbol{\gamma}) = n$.

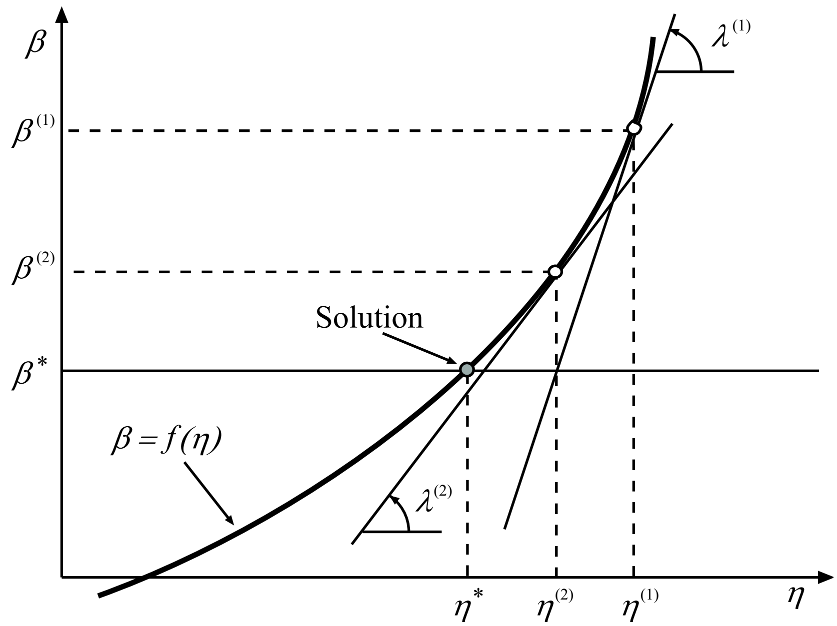

Figure 1: Graphical illustration of the master problem in the single parameter case.

3. Multiple (infinite) solutions: This occurs when $\text{rank}(\mathbf{A}) = \text{rank}\,(\mathbf{A}|\boldsymbol{\gamma}) < n$. The set of all solutions can be obtained by determining the null space of $\mathbf{A}$.

In case of multiple solutions, it is of interest to know the set of all possible local solutions, that is, the differential increments that need to be given to the design variables to have the same set of desired reliability coefficients.

Note that to obtain a unique solution of the problem, it is necessary that the number of design parameters be at least equal to the number of geometric and/or reliability based constraints. But the equality of the number of design parameters and the number of geometric and/or reliability constraints does not guarantee a unique solution.

2.4 Advantages of the Proposed Method

The proposed method has several advantages. These include:

1. The mathematical statement of the problem is simple (only elemental Taylor series expansion and systems of equations need to be dealt with).

2. Standard optimization software can be used without the need of knowing the optimization techniques, that is, the responsibility of finding the solution is given to optimization experts.

3. No critical convergence parameters need to be assessed by the engineer, so that convergence is independent of the right choice of these values.

4. The sensitivities of the reliability indices with respect to the design variables are also readily available without additional costs.

5. The method checks for the existence of no, unique, or multiple solutions, and if multiple solution exists, they are obtained (locally).

2.5 The Proposed Algorithm

The above methodology can be summarized by the following algorithm.

Algorithm 1 (Inverse Reliability Problem)

- **Input:**
  1. *The desired reliability bounds $\boldsymbol{\beta}^*$ with respect to all failure modes.*
  2. *The design variables and parameters.*
  3. *An error value ϵ to control convergence of the procedure.*
- **Output:**
  1. *An indication of whether the problem has one or multiple solutions, and the values of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ associated with one solution of the problem.*
  2. *The sensitivities of the reliability indices β, for all failure modes, with respect to the solution parameters.*
  3. *The set of all local multiple solutions if they exist.*

Initialization. *Initialize the iteration counter to $j = 1$ and select the initial values for the design parameters $\boldsymbol{\eta}^{(j)}$ and $\boldsymbol{\kappa}^{(j)}$ (based on engineering or statistical knowledge and or assumptions).*

Step 1: Solve the subproblems. *Solve problem (19)–(27) to obtain exact values $\alpha_i\beta_i$ for the β-values for all failure modes. The element-wise products $\boldsymbol{\lambda}\cdot\boldsymbol{\alpha}$ and $\boldsymbol{\theta}\cdot\boldsymbol{\alpha}$ are the partial derivatives of the β-values with respect to $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$, respectively. They are proportional to the dual variables $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ corresponding to equations (26) and (27).*

Step 2: Check for existence of solution. *If $rank(\mathbf{A}) \neq rank(\mathbf{A}|\boldsymbol{\gamma})$, stop the process because there is no solution. Otherwise, continue with Step 3.*

Step 3: Solve the master problem. *Update the iteration counter $j = j + 1$, and obtain a new approximation of the unknown parameters solving the linear system of equations (28) in $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$.*

Step 4: Check convergence. *In the current iteration, if changes in the design parameters are larger than a given threshold value ϵ, go to Step 1. Otherwise, go to Step 5.*

Step 5: Check for uniqueness. *If rank*$(\mathbf{A}) = $*rank*$(\mathbf{A}|\boldsymbol{\gamma}) = n$*, then the solution is unique. In this case, return the actual solution* $\boldsymbol{\eta}^*$ *and* $\boldsymbol{\kappa}^*$*, and their corresponding sensitivities* $\boldsymbol{\lambda}^*$ *and* $\boldsymbol{\theta}^*$ *and stop. Otherwise; i.e., if rank*$(\mathbf{A}) = $*rank*$(\mathbf{A}|\boldsymbol{\gamma}) < n$*, continue with Step 6.*

Step 6: Find local multiple solutions. *Obtain the set of all local solutions determined by the null space of* $\mathbf{A}$*, and return the actual solution* $\boldsymbol{\eta}^*$ *and* $\boldsymbol{\kappa}^*$*, and their corresponding sensitivities* $\boldsymbol{\lambda}^*$ *and* $\boldsymbol{\theta}^*$*, together with the null space of matrix* $\mathbf{A}$ *and stop.*

We have implemented the proposed method in GAMS (General Algebraic Modelling System) (see Castillo et al. [23]). GAMS is a software system especially designed for solving optimization problems (linear, non-linear, integer and mixed integer) from small to very large sizes. All the examples have been solved using the generalized reduced gradient method (for more details, see Vanderplaats [24] or Bazaraa, Jarvis and Sherali [25]) that has shown good convergence properties when the variables are constrained. The main advantages of GAMS are:

1. It is a high quality software package (reliable, efficient, fast, widely tested, etc.).
2. It allows the problem to be defined as it is stated mathematically, i.e., without difficult transformations.
3. It allows relations to be handled in implicit or explicit forms.
4. It allows very large (in terms of number of variables or constraints) problems to be solved.

Of course, other optimization programs such as AIMMS [26, 27], AMPL [28, 29], LINDO, What's Best, MPL or the Matlab Optimization Toolbox, can be used instead.

3 Illustrative Numerical Examples

In this section we illustrate the proposed method and compare it with existing methods by means of some numerical examples.

Example 1 (A single limit state function) *Consider the limit state function in the standard normal space*

$$G(\boldsymbol{z}, \eta) = \exp[-\eta(z_1 + 2z_2 + 3z_3)] - z_4 + 1.5, \tag{30}$$

as shown in Der Kiureghian et al. [17] and in Li et al. [18], where the target reliability index is $\beta^* = 2.0$*. Two cases with different hypotheses are analyzed here:*

Case 1: *Considering* η *as a deterministic parameter, using the initial values for* $\eta^{(1)} = 0.15$ *and the variables* $\mathbf{z}^{(0)} = (0.2, 0.2, 0.2, 0.2)^T$*, and a tolerance for convergence* $\epsilon = 10^{-4}$*, the results in Table 1 are obtained. Note that the convergence is achieved in only 4 iterations. In addition, the sensitivity* $\partial\beta/\partial\eta = -0.90847$*, which means that a small increment* $\Delta\eta$ *of* η *produces a decrement in* β *of* $\Delta\beta = 0.90847\Delta\eta$ *of .*

Table 1: Example 1, Case 1: Illustration of the Iterative Process.

j	η	β
1	0.15000	2.28286
2	0.30992	2.05684
3	0.36231	2.00442
4	0.36711	2.00003

Table 2: Example 1, Case 2: Illustration of the Iterative Process.

j	η	β
1	0.20000	2.20417
2	0.33208	2.03821
3	0.37006	2.00217
4	0.37249	2.00001
5	0.37250	2.00000

Case 2: *In this case the parameter is considered as a lognormal random variable with a coefficient of variation* $\kappa = 0.30$. *The mean,* μ, *of this random variable is to be chosen so that* $\beta^* = 2.0$. *Table 2 shows the iterations converge to a unique solution. In this case, the sensitivities are:*

$$\frac{\partial \beta}{\partial \mu} = -0.88805 \quad \textit{and} \quad \frac{\partial \beta}{\partial \kappa} = 0.03081.$$

Thus, the reliability index is more sensitive to changes in μ *than to changes in* κ *and it decreases with* μ *but increases with* κ.

Example 2 (Three limit state functions) *Consider the following three limit state functions related to four random variables:*

$$g_1(\boldsymbol{x}) = x_1^2 - 4x_2 - 2x_3x_4, \tag{31}$$

$$g_2(\boldsymbol{x}) = 2x_1x_4 - x_2x_3, \tag{32}$$

$$g_3(\boldsymbol{x}) = x_1x_2x_4 - 2x_3, \tag{33}$$

with given target reliability indices $\boldsymbol{\beta}^* = (3.0, 3.5, 4.0)^T$. *We consider three cases:*

Case 1 *The design parameters are the mean values* η_1, η_2, *and* η_3 *of three of these variables, and the distributional assumptions are shown in Table 3. The initial values of the design parameters are* $(5, 2, 2)$. *Table 4 shows the convergence of the process is attained in* 4 *iterations and Table 5 gives the sensitivities. Note that the sensitivities of the* β*-values with respect to the* κ*-values (coefficients of variation) are negative, thus increasing the coefficients of variation will decrease the reliability indices, which implies that the probabilities of failure will increase with* κ.

Table 3: Example 2, Case 1: Design Variables and Distributional Assumptions.

Variable	Mean Value	Coefficient of Variation	Distribution Type
x_1	η_1	$\kappa_1 = 0.01$	Normal
x_2	η_2	$\kappa_2 = 0.2$	Lognormal
x_3	η_3	$\kappa_3 = 0.1$	Lognormal
x_4	$\eta_4 = 1$	$\kappa_4 = 0.1$	Gumbel

Table 4: Example 2, Case 1: Illustration of the Iterative Process.

j	$\eta_1^{(j)}$	$\eta_2^{(j)}$	$\eta_3^{(j)}$	$\beta_1^{(j)}$	$\beta_2^{(j)}$	$\beta_3^{(j)}$
1	5.00000	2.00000	2.00000	4.90903	3.92591	3.75717
2	4.31505	2.15542	1.76851	2.88599	3.49957	3.97454
3	4.36348	2.16168	1.78304	2.99919	3.49989	3.99986
4	4.36379	2.16169	1.78312	3.00000	3.50000	4.00000

Case 2 *To show how the method detects the case of infinite solutions, we take all four parameters $\eta_1, \eta_2, \eta_3, \eta_4$ in Table 3 to be the design values, thus we have four parameters and three reliability conditions. The solutions given by GAMS are:*

$$\eta_1 = 4.557, \quad \eta_2 = 2.1617, \quad \eta_3 = 2.1021, \quad \eta_4 = 1.1772,$$

shown in Table 6, and the general local solution is:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} 4.5570 \\ 2.1617 \\ 2.1021 \\ 1.1772 \end{pmatrix} + \rho \begin{pmatrix} 1.2300 \\ 0 \\ 2.4538 \\ 1 \end{pmatrix}; \quad \rho \in \mathbb{R},$$

which shows the required local increments of the design variables to attain the desired reliability indices values. This means that increasing η_1, η_3, and η_4 by 1.23ρ, 2.4538ρ, and ρ, respectively, for small values of ρ and no changes in η_2, gives to the same values of the reliability indices.

Case 3 *In this case the standard deviation of variable X_1 and the mean values of X_2 and X_3 are taken as design parameters and the distributional assumptions are shown in Table 7. We consider here the case where all variables are uncorrelated. Table 8 shows the convergence of the process and Table 9 gives the sensitivities, where the relative influences of the η and κ parameters on the reliability indices can be observed.*

Case 4 *In the above case, the variables are assumed to be uncorrelated. To show the effect of correlation, we consider the case where the variables X_1 and X_2 have a correlation*

Table 5: Example 2, Case 1: Sensitivities of the β-Values with Respect to the Parameters.

x	$\partial\beta_1/\partial x$	$\partial\beta_2/\partial x$	$\partial\beta_3/\partial x$
η_1	2.82100	0.98424	0.98666
η_2	–2.28247	–1.98689	1.99176
η_3	–0.68482	–2.40872	–2.41463
η_4	–1.22113	4.29504	4.30557
κ_1	–4.58017	–0.64761	–0.74408
κ_2	–13.09599	–11.59053	–15.08793
κ_3	–0.32201	–5.96740	–6.91545
κ_4	–0.22688	–4.57130	–4.98348

Table 6: Example 2, Case 2: Illustration of the Iterative Process.

ν	$\eta_1^{(\nu)}$	$\eta_2^{(\nu)}$	$\eta_3^{(\nu)}$	$\eta_4^{(\nu)}$	$\beta_1^{(\nu)}$	$\beta_2^{(\nu)}$	$\beta_3^{(\nu)}$
1	5.00000	2.00000	2.00000	1.00000	4.90903	3.92591	3.75717
2	4.49679	2.15542	2.21765	1.18822	2.79700	3.44547	3.92032
3	4.55613	2.16169	2.19236	1.17754	2.99681	3.49976	3.99973
4	4.55700	2.16169	2.19211	1.17724	3.00000	3.50000	4.00000

Table 7: Example 2, Case 3: Design Variables and Distributional Assumptions.

Variable	Mean Value	Coefficient of Variation	Distribution Type
x_1	$\eta_1 = 6$	κ_1	Normal
x_2	η_2	$\kappa_2 = 0.2$	Lognormal
x_3	η_3	$\kappa_3 = 0.1$	Lognormal
x_4	$\eta_4 = 1$	$\kappa_4 = 0.1$	Gumbel

coefficient $\kappa_5 = 0.8$. *Table 10 shows the convergence of the process and Table 11 gives the sensitivities. It can be seen that the presence of correlation causes substantial changes in the design parameters.*

Example 3 (Sadovský example) *Consider the following limit state function in the standard normal space, as in Sadovský [20].*

$$G(\boldsymbol{z}) = \frac{1}{2}\sum_{i=1}^{4} \kappa_i z_i + \beta - \eta z_5, \tag{34}$$

where the specified reliability index is $\beta^* = 4.0$ *and the principal curvatures are* $\kappa_i = 0.8 - 0.2(i-1);\ i = 1,\ldots,4$. *The initial values for the variables are* $\boldsymbol{z}^{(0)} = (0.1, 0.1, 0.1, 0.1, 0.1)^T$

Table 8: Example 2, Case 3: Illustration of the Iterative Process.

j	$\eta_2^{(j)}$	$\eta_3^{(j)}$	$\kappa_1^{(j)}$	$\beta_1^{(j)}$	$\beta_2^{(j)}$	$\beta_3^{(j)}$
1	3.00000	3.00000	0.60000	2.65851	1.14061	4.10454
2	2.13744	1.98133	0.79952	2.96971	3.69678	4.00158
3	2.19721	2.08110	0.76742	3.00126	3.49551	3.99929
4	2.19614	2.07856	0.76826	3.00000	3.50000	4.00000
5	2.19614	2.07855	0.76826	3.00000	3.50000	4.00000

Table 9: Example 2, Case 3: Sensitivities of the β-Values with Respect to the Parameters.

x	$\partial\beta_1/\partial x$	$\partial\beta_2/\partial x$	$\partial\beta_3/\partial x$
η_1	1.22232	0.78095	0.81981
η_2	–0.76072	–1.55992	1.51727
η_3	–0.32405	–1.64817	–1.60310
η_4	–0.67355	3.42580	3.33213
κ_1	–3.44349	–1.63991	–2.06535
κ_2	–1.28896	–7.24050	–9.18161
κ_3	–0.06807	–3.72778	–4.06734
κ_4	–0.01079	–3.16512	–3.29011

Table 10: Example 2, Case 4: Illustration of the Iterative Process.

j	$\eta_2^{(j)}$	$\eta_3^{(j)}$	$\kappa_1^{(j)}$	$\beta_1^{(j)}$	$\beta_2^{(j)}$	$\beta_3^{(j)}$
1	3.00000	3.00000	0.60000	3.99969	1.58532	3.26436
2	3.08635	1.90852	0.81251	3.23005	4.11441	3.99731
3	3.28513	1.99117	0.82850	3.00504	3.51452	3.99709
4	3.29190	1.99206	0.82868	3.00000	3.50001	3.99999

and $\eta^{(0)} = 0.15$. *Table 12 shows the convergence of the process. In addition, the sensitivity is* $\partial\beta/\partial\eta = -4$.

Example 4 (A limit state function from Kim et al.) *In this example, a limit state function from Kim et al. [30] is considered:*

$$g(\boldsymbol{x}) = \exp((x_1 + 2) + 6.2) - \exp(0.3x_2 + x_3) - 200, \tag{35}$$

where X_1 *and* X_2 *are standard normal and* X_3*, assumed normal, is treated as a design variable with coefficient of variation* $\kappa_3 = 0.1$ *and unknown mean value, the design parameter* η_3*. The target reliability index is* $\beta^* = 2.5$*. Table 13 shows the convergence of the process*

Table 11: Example 2, Case 4: Sensitivities of the β-Values with Respect to the Parameters.

x	$\partial\beta_1/\partial x$	$\partial\beta_2/\partial x$	$\partial\beta_3/\partial x$
η_1	1.52529	0.89235	0.72172
η_2	–0.56874	–1.77144	0.63872
η_3	–0.43651	–2.92732	–1.05549
η_4	–0.86956	5.83141	2.10261
κ_1	–4.42652	0.57598	–2.68825
κ_2	3.85286	–10.04233	–7.71261
κ_3	–0.13850	–11.20670	–1.54269
κ_4	–0.06466	–7.48251	–1.55252
κ_5	1.40599	2.98899	–0.99616

Table 12: Example 3: Illustration of the Iterative Process.

j	$\eta^{(j)}$	$\beta^{(j)}$
1	0.15000	26.66667
2	0.27750	14.41441
3	0.47799	8.36831
4	0.72751	5.49821
5	0.92575	4.32083
6	0.99449	4.02218
7	0.99997	4.00012
8	1.00000	4.00000

Table 13: Example 4: Illustration of the Iterative Process.

j	$\eta_3^{(j)}$	$\beta^{(j)}$
1	4.00000	3.45532
2	5.05935	2.18575
3	4.82430	2.49808
4	4.82284	2.50000

that is attained in 4 *iterations and Table 14 gives the sensitivities. Note that the sensitivities of the reliability indices to coefficients of variations ($\boldsymbol{\kappa}$) are negative, that is, the larger the coefficient of variation, the smaller the reliability indices.*

Table 14: Example 4: Sensitivities of the β-Values with Respect to the Parameters.

x	$\partial\beta/\partial x$
η_1	0.75478
η_2	–0.34648
η_3	–1.31575
κ_1	–1.42425
κ_2	–0.30012
κ_3	–7.75633

4 Extensions

If, as indicated previously, the equalities in (10) are relaxed, replacing them by inequalities as in (5) and adding constraints (9), then the resulting problem becomes more realistic from the engineering point of view, and can be solved using a similar approach. Then, the master problem becomes

$$\begin{aligned} \beta_i^* &\leq \beta_i^{(j)} + \sum_{\forall m} \frac{\partial f_i(\boldsymbol{\eta},\boldsymbol{\kappa})}{\partial \eta_m}\left(\eta_m - \eta_m^{(j)}\right) + \sum_{\forall l} \frac{\partial f_i(\boldsymbol{\eta},\boldsymbol{\kappa})}{\partial \kappa_l}\left(\kappa_l - \kappa_l^{(j)}\right), && (36)\\ h_j(\boldsymbol{\eta},\boldsymbol{\kappa}) &\leq 0;\; j = 1,2,\ldots,m. && (37) \end{aligned}$$

The most general solution, if constraints (37) are linear (typical for variable bounds), is a polyhedron, i.e., the sum of a polytope (set of vectors generated by linear convex combinations of a set of vectors), a cone (set of vectors generated by non-negative linear combinations of a set of vectors), and a linear space (set of vectors generated by linear combinations of a set of vectors) (see Castillo et al. [31, 23]).

In some cases, for example when the number of design parameters exceeds the number of constraints, the solution may be not unique, but the uniqueness can be enforced by introducing the optimization of a related objective function (see Castillo et al. [32, 33, 34, 35, 36]). In this case the master problem becomes:

$$\beta = \underset{\boldsymbol{\eta},\boldsymbol{\kappa}}{\text{Minimize}} \quad c(\boldsymbol{\eta},\boldsymbol{\kappa})$$

subject to

$$\begin{aligned} \beta_i^* &\leq \beta_i^{(j)} + \sum_{\forall m} \frac{\partial f_i(\boldsymbol{\eta},\boldsymbol{\kappa})}{\partial \eta_m}\left(\eta_m - \eta_m^{(j)}\right) + \sum_{\forall l} \frac{\partial f_i(\boldsymbol{\eta},\boldsymbol{\kappa})}{\partial \kappa_l}\left(\kappa_l - \kappa_l^{(j)}\right), && (38)\\ h_j(\boldsymbol{\eta},\boldsymbol{\kappa}) &\leq 0;\; j = 1,2,\ldots,m, && (39) \end{aligned}$$

where $c(\boldsymbol{\eta},\boldsymbol{\kappa})$ is a cost, or similar, function.

Thus, the generalized inverse reliability problem is a special case of a more general reliability-based optimization problem with multiple design parameters, constraints, and a cost function (see Thoft-Christensen and Murotsu [37]), and arises when one is seeking

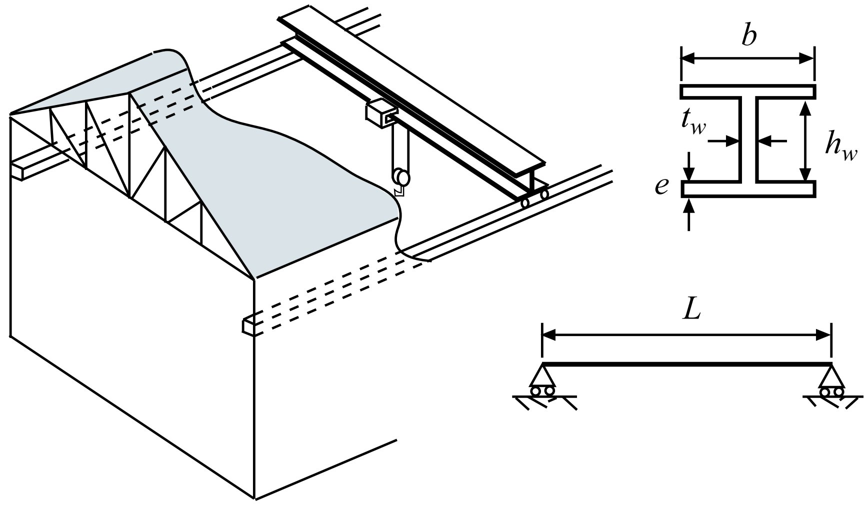

Figure 2: An illustration of a bridge crane.

directly the values of some parameters related either to the limit state functions ($\boldsymbol{\eta}$) or to the statistical assumptions ($\boldsymbol{\kappa}$) corresponding to specified reliability levels. This problem arises in a large number of engineering applications. An illustrative example is shown in the following section.

5 Bridge Crane Design Application

Modern industries require equipment for handling large, heavy, or bulky objects. To meet this demand engineers specialize in overhead material handling equipments such as bridge cranes, hoists, and monorails. In this section we apply the extension of the proposed method introducing the optimization of a related objective function to an engineering example: the design of a bridge crane.

Figure 2 depicts an under running overhead crane with a single girder. Its structural elements must be manufactured in accordance with current mandatory requirements of the National Safety and Health Act, OSHA Sections 1910.179 and 1910.309 as applicable. Additionally, all ACECO cranes are manufactured in accordance with the appropriate standard of ANSI specifications, the National Electric Code, and the Crane Manufacturers Association of America (CMAA) specifications.

Crane girders are designed using structural steel beams (reinforced as necessary) or fabricated plate box sections. Bridge girders are designed for loadings, stresses, and stability in accordance with current CMAA design specifications. We now apply the engineering design method developed in Section 3 to the design of an overhead crane in Figure 2. In particular, the bridge girder dimensions that allow trolley travelling horizontally are calculated. It consists of a box section fabricated from plate of structural steel, for the web, top and bottom plates, so as to provide for maximum strength at minimum dead weight. Maximum allowable vertical girder deflection is a function of span length.

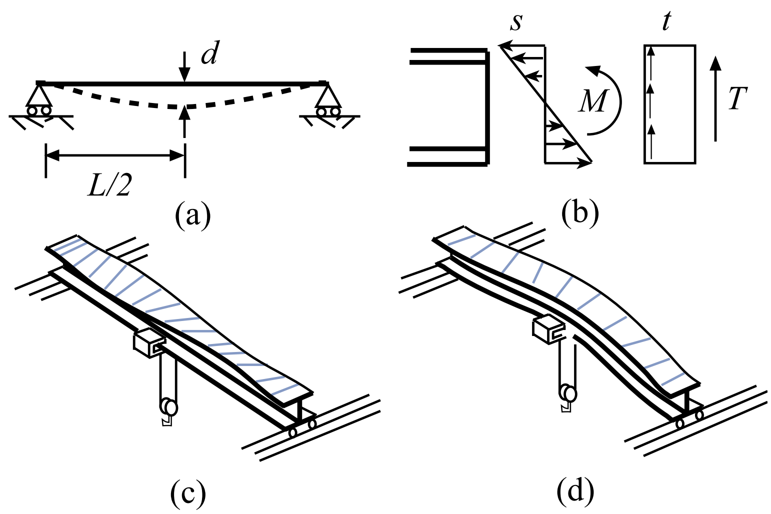

Figure 3: Illustrations of the bridge girder modes of failure.

5.1 Design Variables and Parameters

Consider the girder and the cross-section shown in Figure 2, where L is the span or distance from centerline to centerline of runway rails; b and e are the flange width and thickness, respectively; and t_w and h_w are the web thickness and height, respectively. As indicated in Section 1, the set of variables involved in the problem can be partitioned into four subsets:

1. *Statistical variables*: $\boldsymbol{x} = \{b, e, t_w, h_w, f_y, E, \nu, \gamma_y, L, P\}$, where f_y is the value of the elastic limit of structural steel, E is the Young modulus of the steel, ν is the Poisson modulus, γ_y is the steel unit weight, L is the span length, and P is the maximum load supported by the girder.

2. *Design parameters*: $\boldsymbol{\eta} = \{\mu_b, \mu_e, \mu_{t_w}, \mu_{h_w}, \mu_{f_y}, \mu_E, \mu_\nu, \mu_{\gamma_y}, \mu_L\}$ are the mean values of the corresponding statistical variables. Instead of using mean values, other representative values of the variables involved can be used (i. e., characteristic values). Note that in this case all the design parameters are considered random, this is the most general case.

3. *Random model parameters*: $\boldsymbol{\kappa} = \{\lambda_P, \delta_P, v_{f_y}, v_E, v_\nu, v_{\gamma_y}, v_L, v_d\}$, where v refers to the coefficient of variation of the corresponding variable, λ_P and δ_P are the Gumbel distribution model parameters for the maximum load, and v_d is the coefficient of variation of the design variables corresponding to the cross-section dimensions.

4. *Intermediate variables*: $\boldsymbol{\psi} = \{W, I_{xx}, I_{yy}, I_t, G, \sigma, \tau, M_{cr}, \delta, M, T\}$. These variables simplify the statement of the problem.

Assume that the following four failure modes are considered (see Figure 3):

1. *Maximum allowed deflection.* The maximum deflection constraint (see Figure 3(a)) is defined as the ratio:

$$g_d(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \frac{\delta_{max}}{\delta}, \tag{40}$$

where δ is the maximum deflection on the center of the girder and δ_{max} is the maximum deflection allowed by codes.

2. *Damage limit state of the steel upper and lower flanges.* The ratio of the actual strength to actual stresses:

$$g_u(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \frac{f_y}{\sqrt{\sigma^2 + 3\tau^2}}, \tag{41}$$

is the limit state constraint, where σ and τ are the normal and tangential stresses at the center of the beam, respectively (see Figure 3(b)).

3. *Damage limit state of the steel web.* The bearing capacity limit state is the ratio of the shear strength capacity to actual shear stress at the center of the beam (see Figure 3(b)):

$$g_w(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \frac{f_y}{\sqrt{3}\tau}. \tag{42}$$

4. *Global Buckling.* The global buckling limit state equation is the ratio of the maximum moment applied at the center of the beam to the critical moment against buckling of the cross section (see Figure 3(d)):

$$g_b(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \frac{M}{M_{cr}}. \tag{43}$$

The bridge girder is safe if and only if g_d, g_u, g_w and $g_b \geq 1$.

5.2 Design Constraints

The following constraints are considered:

1. *Geometrical and mechanical properties of the girder.* The moments of inertia I_{xx} and I_{yy} are obtained as

$$I_{xx} = \frac{1}{12}\left(b(h_w + 2e)^3 - (b - t_w)h_w^3\right), \tag{44}$$

and

$$I_{yy} = \frac{1}{12}\left(2eb^3 + h_w t_w^3\right), \tag{45}$$

whereas the torsional moment of inertia is obtained using

$$I_t = \frac{1}{3}\left(2be^3 + h_w t_w^3\right). \tag{46}$$

The deflection at the center of the beam is calculated using:

$$\delta = \frac{PL^3}{48EI_{xx}} + \frac{5WL^4}{384EI_{xx}}, \tag{47}$$

where

$$W = \gamma_s(2eb + t_w h_w), \tag{48}$$

is the girder bridge weight per unit length. The stresses at the center of the beam are calculated as:

$$\begin{aligned} T &= P/2, && (49)\\ M &= PL/4, && (50) \end{aligned}$$

where T and M are the shear force and moment, respectively. Thus,

$$\sigma = \frac{M(h_w + e)}{2I_{xx}}, \tag{51}$$

and

$$\tau = \frac{T}{h_w t_w}. \tag{52}$$

The critical moment for global buckling is

$$M_{cr} = \frac{\pi}{L}\sqrt{EGI_{yy}I_t} \tag{53}$$

with auxiliary parameter

$$G = \frac{E}{2(1+\nu)}.$$

2. *Code and other constraints.* The following constraints are fixed by the codes and other requirements. These correspond to constraints (37):

$$\begin{aligned} 0.008 \le \quad \mu_e \quad &\le 0.038\ (m), && (54)\\ 0.008 \le \quad \mu_{t_w} \quad &\le 0.038\ (m), && (55)\\ 0.3 \le \quad \mu_b \quad &\le 1.0\ (m), && (56)\\ 0.5 \le \quad \mu_{h_w} \quad &\le 1.5\ (m). && (57) \end{aligned}$$

Thus, for example, to support the trolley, the unit that travels on the bottom flange of the bridge girder and carries the hoist, the minimum flange width (b) must be 0.30 meters.

The maximum deflection allowed is

$$\delta_{max} = L/888.$$

To avoid local buckling (see Figure 3(c)) the design must satisfy the following constraint:

$$\frac{\mu_b}{2\mu_e} \le 15\sqrt{\frac{276000}{\mu_{f_y}}}, \tag{58}$$

where μ_{f_y} is the mean steel strength in kN/m^2.

5.3 A Numerical Example

To perform a probabilistic design in the bridge girder example, the joint probability density of all variables is required. The statistical distributions of the variables involved ($\boldsymbol{x}$) are taken from the Probabilistic Model Code [38, 39]:

1. The maximum supported load has a Gumbel (Maximum) distribution, $G(P; \lambda_P, \delta_P)$.

2. The variables f_y, E and ν have log-normal distributions:

$$\begin{array}{lcl} \log f_y & \sim & N(\mu_{f_y}, \mu_{f_y} v_{f_y}), \\ \log E & \sim & N(\mu_E, \mu_E v_E), \\ \log \nu & \sim & N(\mu_\nu, \mu_\nu v_\nu), \end{array} \tag{59}$$

   where $\mu_{f_y} = 355{,}000\ kN/m^2$, $\mu_E = 210{,}000{,}000\ kN/m^2$, $\mu_\nu = 0.3$, $v_E = 0.03$ and $v_\nu = 0.03$. The coefficient of variation v_{f_y} is unknown and has to be selected by the proposed method.

3. The variables L, γ_y, b, e, t_w and h_w have the following normal distributions:

$$\begin{array}{lll} L \sim N(\mu_L, \mu_L v_L), & \gamma_y \sim N(\mu_{\gamma_y}, \mu_{\gamma_y} v_{\gamma_y}), & b \sim N(\mu_b, \mu_b v_d), \\ e \sim N(\mu_e, \mu_e v_d), & t_w \sim N(\mu_{t_w}, \mu_{t_w} v_d), & h_w \sim N(\mu_{h_w}, \mu_{h_w} v_d), \end{array} \tag{60}$$

   where

$$\lambda_P = 600\ kN, \quad \delta_P = 70.2\ kN, \quad \mu_L = 6\ m, \quad \mu_{\gamma_y} = 78.5\ kN/m^3.$$

   The means of the design variables b, e, t_w, h_w and the variation coefficients v_L and v_d are the values obtained from the inverse reliability problem.

In addition, the following constraints are used to bound the relevant variables and to enforce reasonable values on the desired parameters:

$$0.01 \le v_{f_y} \le 0.2, \tag{61}$$

$$0.01 \le v_L \le 0.3, \tag{62}$$

$$0.01 \le v_d \le 0.3. \tag{63}$$

These correspond to constraints (37).

The target reliability indices are:

$$\beta_d^* = 2.05, \quad \beta_u^* = 3.719, \quad \beta_w^* = 3.719, \quad \beta_b^* = 3.29,$$

which correspond approximately to probabilities of failure of 0.02, 0.0001, 0.0001, and 0.0005, respectively. Note that violations of limit states with more serious consequences are associated with higher reliability indices, but other reasonable values could have been chosen.

Using the Rosenblatt [14] transformation, the set of random variables is transformed into a set of standard normal random variables $Z_1, Z_2, \cdots, Z_{10}$ by

$$
\begin{aligned}
&\Phi(z_1) = \exp\left\{-\exp\left[-\frac{(P-\lambda_P)}{\delta_P}\right]\right\}, && z_2 = \frac{\log f_y - \log\left(\mu_{f_y}/\sqrt{1+v_{f_y}^2}\right)}{\sqrt{\log\left(1+v_{f_y}^2\right)}},\\
&z_3 = \frac{\log E - \log\left(\mu_E/\sqrt{1+v_E^2}\right)}{\sqrt{\log\left(1+v_E^2\right)}}, && z_4 = \frac{\log \nu - \log\left(\mu_\nu/\sqrt{1+v_\nu^2}\right)}{\sqrt{\log\left(1+v_\nu^2\right)}},\\
&z_5 = \frac{\gamma_y - \mu_{\gamma_y}}{\mu_{\gamma_y} v_{\gamma_y}}, && z_6 = \frac{L-\mu_L}{\mu_L v_L},\\
&z_7 = \frac{b-\mu_b}{\mu_b v_d}, && z_8 = \frac{e-\mu_e}{\mu_e v_d},\\
&z_9 = \frac{t_w-\mu_{t_w}}{\mu_{t_w} v_d}, && z_{10} = \frac{h_w-\mu_{h_w}}{\mu_{h_w} v_d},
\end{aligned}
\tag{64}
$$

where the coefficient of variation of e, b, t_w, h_w, is assumed to be equal to v_d.

To extend the inverse reliability problem to reliability optimization, a cost function related to the unknown parameters has to be defined. In this case, we minimize the direct cost of the bridge crane using a penalty factor to consider the cost of reducing variation coefficients,

$$
c(\boldsymbol{\eta},\boldsymbol{\kappa}) = c_y \mu_{\gamma_s}\mu_L(2\mu_e\mu_b + \mu_{t_w}\mu_{h_w})\left(1+\exp(-30v_{f_y})/2 + \exp(-30v_L)/2 + \exp(-30v_d)/2\right), \tag{65}
$$

where $c_y = 0.24$ ($\$/N$), is the cost per unit weight of steel, $\mu_{\gamma_s}\mu_L(2\mu_e\mu_b + \mu_{t_w}\mu_{h_w})$ is the weight of the bridge crane and the factor involving the variation coefficients v_{f_y}, v_L, and v_d considers the cost increase due to variability of the corresponding variables. Note that increasing the random variability (coefficients of variation) implies reducing the execution or control costs and vice versa. This cost function has been selected for illustration purposes, but other solutions could have been chosen.

From the above presentation, the inverse reliability problem in this case can now be stated as minimizing the cost function (65) with respect the parameters $\{\mu_b, \mu_e, \mu_{t_w}, \mu_{h_w}, v_{f_y}, v_L, v_d\}$ (the remaining parameters of $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ are taken as fixed values using additional constraints) in such a way that the corresponding reliability indices are greater or equal to the target ones $\boldsymbol{\beta}^*$, subject to equations (40)–(43) defining the limit state equations related to the different failure modes, (44)–(58) imposing code and other requirements, and (61)–(63) leading to reasonable values for the desired parameters.

The results of the proposed method are given in Table 15. It can be seen that convergence is reached after 14 iterations but the values of the parameters do not change significantly after iteration 6. The first iteration column shows the values of the design variables, and the actual reliability coefficients (β-values) associated with the initial design values. Note that the β's do not hold. Then, the iterative process continues increasing the cost until all constraints hold. The last column of the table shows the values of the design variables together with the final β-values, where it can be seen that they are equal to the target reliability level except for that corresponding to steel web, this implies that satisfaction of the other failure modes in terms of safety ensures the bridge crane to be safe enough against web damage.

Table 15: Bridge Crane Design: Illustration of the Iterative Process.

j	1	2	3	4	5	6	14
$cost$	3243.1	3406.6	3370.6	3371.6	3371.6	3371.8	3371.8
$\mu_b^{(j)}$	0.35000	0.52880	0.51749	0.51676	0.51552	0.51675	0.51626
$\mu_e^{(j)}$	0.02000	0.01999	0.01956	0.01954	0.01949	0.01954	0.01952
$\mu_{t_w}^{(j)}$	0.00800	0.00800	0.00800	0.00800	0.00800	0.00800	0.00800
$\mu_{h_w}^{(j)}$	1.00000	0.76898	0.77335	0.76830	0.76873	0.76866	0.76867
$v_{f_y}^{(\nu)}$	0.15000	0.16866	0.15387	0.14255	0.14282	0.14265	0.14269
$v_L^{(\nu)}$	0.10000	0.11970	0.08493	0.08075	0.08288	0.08230	0.08241
$v_d^{(\nu)}$	0.10000	0.08941	0.05502	0.06072	0.05790	0.05980	0.05905
$\beta_u^{(\nu)}$	3.46043	3.36588	3.68875	3.71432	3.71849	3.71877	3.71900
$\beta_w^{(\nu)}$	5.72148	4.89613	5.41711	5.46030	5.48699	5.47003	5.47701
$\beta_b^{(\nu)}$	0.59192	2.72843	3.37080	3.28561	3.28895	3.28898	3.29013
$\beta_d^{(\nu)}$	2.16990	1.66615	2.13351	2.04748	2.04963	2.04926	2.05014

The method also gives the sensitivities associated with the β-values, that are shown in Table 16. It is useful to know how much the β-values change due to a small change in a single data value (e.g., the means or the coefficients of variation). In this table the designer can easily analyze how the quality of the material (reduced standard deviation of f_y) or precision in the applied loads (reduced standard deviation of P) influences the safety of the beam. Note that an increase in the dispersion (standard deviation or coefficient of variation) leads to a decrease of the β indices.

6 Conclusions

The method presented in this paper takes full advantage of the optimization frameworks and allows practitioner engineers to solve the inverse reliability problem with either single or multiple design parameters, treated as either deterministic or random. In addition, sensitivity analysis can be easily performed by transforming the input parameters into artificial variables, which are constrained to take their associated constant values. The provided examples illustrate how the proposed procedure can be applied to a wide range of practical engineering design problems. In some applications, particularly in multiple design variable problems, the procedure provides a cost-saving alternative to applying and interpolating results from the standard methods. Additional advantages of the proposed method include:

1. The method is simple and allows and easy connection with optimization frameworks.

2. The reliability analysis takes full advantage of the optimization packages, which allows the solution of huge problems without the need of being an expert in optimization techniques.

Table 16: Bridge Crane Design: Sensitivities of the β-Values with Respect to the Parameters.

x	$\partial\beta_u/\partial x$	$\partial\beta_w/\partial x$	$\partial\beta_b/\partial x$	$\partial\beta_d/\partial x$
μ_b	4.185	0.000	12.566	6.431
μ_e	110.376	0.000	321.319	178.958
μ_{t_w}	181.396	450.358	40.683	38.913
μ_{h_w}	4.975	4.687	0.141	9.729
μ_{f_y}	0.000	0.000	0.000	0.000
μ_E	0.000	0.000	0.000	0.000
μ_ν	0.000	0.000	–0.066	0.000
μ_{γ_y}	0.000	0.000	0.000	0.000
μ_L	–0.394	0.000	–1.099	–1.230
λ_P	–0.003	–0.003	–0.004	–0.005
δ_P	–0.022	–0.030	–0.015	–0.011
υ_{f_y}	–7.290	–10.446	0.000	0.000
υ_E	0.002	0.000	–0.007	–0.012
υ_ν	0.000	0.000	0.001	0.000
υ_{γ_y}	0.000	0.000	0.000	0.000
υ_L	–1.538	0.000	–9.233	–7.686
υ_d	–6.224	–9.799	–18.693	–10.578

3. The responsibility for iterative methods is given to the optimization software.

4. Constraints can be written in either implicit or explicit form.

5. Unlike level II methods (FORM), the proposed method does not need to invert the Rosenblatt transformation and the failure region need not be written in terms of the normalized transformed variables.

6. Sensitivity values with respect to the target reliability levels are given, without additional cost, by the values of the dual problem.

7. The extension to reliability-based optimization is easy.

8. It can be applied to different types of problems such as linear, non-linear, and mixed-integer problems. The designer needs just to choose the adequate optimization algorithm.

9. The method checks for the existence of no, unique, or multiple solutions.

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