Reliability and Decomposition Techniques to Solve Certain Class of Stochastic Programming Problems

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Abstract

Reliability based techniques has been an area of active research in structural design during the last decade, and different methods have been developed. The same has occurred with stochastic programming, which is a framework for modeling optimization problems involving uncertainty. The discipline of stochastic programming has grown and broadened to cover a wide range of applications, such as agriculture, capacity planning, energy, finance, fisheries management, production control, scheduling, transportation, water management, etc., and because of this, techniques for solving stochastic programming models are of great interest for the scientific community. This paper presents a new approach for solving a certain type of stochastic programming problems presenting the following characteristics: i) the joint probability distributions of random variables are given, ii) these do not depend on the decisions made, and iii) random variables only affect the objective function. The method is based on mathematical programming decomposition procedures and first-order reliability methods, and constitutes an efficient method for optimizing quantiles in high-dimensional settings. The solution provided by the method allows us to make informed decisions accounting for uncertainty.

Keywords: Decision analysis, Reliability analysis, Risk metric, Stochastic programming, Uncertainty

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1. Introduction

The subject of structural reliability, which has been an intensive research area over the last decade, offers a rational framework to quantify uncertainties mathematically. It combines theories of probability, random variables and random processes, and it has result in the development of different methods to deal with uncertainty, such us, First-Order Reliability Methods (FORM). On the other hand, stochastic programming is a framework for modeling optimization problems that involve uncertainty. Since real world problems almost invariably include uncertain parameters, the discipline of stochastic programming has grown and broadened to cover a wide range of applications. Stochastic programming models take the advantage of the fact that probability distributions governing the data are known or can be estimated, which allows decision makers to maximize some function of the decision and random variables.

This paper proposes a new method to solve certain class of stochastic programming problems based on FORM and mathematical programming decomposition techniques.

Between the different applied and studied stochastic programming models, in this paper we focus on a certain class of problems with the following characteristics related to the random variables:

1. Their joint probability distribution is given or can be estimated parametrically.
2. Their distributions do not depend on the decisions variables.
3. The random variables only affect the objective function.

This model structure is suitable for managing profit risks in energy trading, where energy agents need appropriate methodologies to establish the most profitable strategy for a specified risk level. In this particular case the risk involved is due to the stochastic nature of energy prices. Additionally, this problem structure can be exploited by investment companies, brokerage funds, or any business participating in financial markets, and also to the optimization of percentiles in contexts outside finance or markets.

The mathematical structure of the studied stochastic programming model can be defined as follows:

$$\underset{\boldsymbol{x}}{\text{Maximize}} \quad q\left(f(\boldsymbol{x}, \boldsymbol{y})\right) \ , \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{x}) &= \mathbf{0} && (2) \\ \boldsymbol{g}(\boldsymbol{x}) &\leq \mathbf{0}, && (3) \end{aligned}$$

where $\boldsymbol{x}$ is the vector including the decision variables representing what we may generally call portfolio, $\boldsymbol{y}$ is the stochastic variable vector including all random variables involved, $f(\boldsymbol{x}, \boldsymbol{y})$ is the objective function, i.e., loss, cost, profit or utility, and $\boldsymbol{h}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are the constraints that must be satisfied, i.e., maximum or minimum operational limits, market constraints, etc.

Note that due to the stochastic nature of $\boldsymbol{y}$ the objective function $f(\boldsymbol{x}, \boldsymbol{y})$ is also a random variable whose probability distribution function depends on the probability distribution functions of the random variables $\boldsymbol{y}$, and the values of the decision variables $\boldsymbol{x}$. And since a random variable cannot be maximized, the objective function is replaced by a quantile of the objective function $q\left(f(\boldsymbol{x}, \boldsymbol{y})\right)$.

Hereafter, we use uppercase letters ($\boldsymbol{Y}$) to refer to random variables, and lowercase letters ($\boldsymbol{y}$) to refer to particular instances of these variables.

This paper provides a methodology to solve problem (1)-(3) using First-Order Reliability Methods (FORM) and mathematical programming decomposition techniques.

The solution is achieved by means of an iterative scheme, which involves two procedures, namely, decision making and quantile evaluation:

1. **Decision making:** For given values of the random variables $\boldsymbol{y}$, the decision variables $\boldsymbol{x}^*$ which maximizes the objective function are obtained.
2. **Quantile evaluation:** For the decisions made in the previous step, FORM is used to update the values of the random variables required to achieve the target quantile. Note that FORM involves also solving a mathematical programming problem.

This process is repeated until the quantile associated with the decision variables coincides with the target quantile.

In the last decade, the advances in mathematical programming techniques that allow the solution of large complicated problems have permitted researchers to state and develop new methods for optimal risk-informed decision making. Since measures of risk have a crucial role in optimization under uncertainty several metrics and methods have been proposed in the literature.

The simplest method consists of solving problem (1)-(3) by replacing the random variables with their expected values ($\bar{\boldsymbol{y}}$), usually based on historical observations. The optimal solution of this problem is a central value of the objective function, which could also coincide with the expected value, i.e. $\mu_F = f(\boldsymbol{x}, \bar{\boldsymbol{y}}) = \mathrm{E}[f(\boldsymbol{x}, \boldsymbol{Y})]$. For example, this method is used in [1] within the context of energy markets. The main drawback of this method is that it does not take into account the volatility of any of the variables and may be highly risky. Since the pioneering work of Markowitz [2], and in order to account for volatility, several authors, such as, Conejo et al. [3], replace the objective function (1) with:

$$\underset{\boldsymbol{x}}{\text{Maximize}} \quad E[f(\boldsymbol{x}, \boldsymbol{Y})] - \delta V[f(\boldsymbol{x}, \boldsymbol{Y})] = \mu_F - \delta \sigma_F^2 \ , \tag{4}$$

where μ_F and σ_F^2 are the mean and the variance of the objective function random variable $f(\boldsymbol{x}, \boldsymbol{y})$, and the parameter δ is a weighting constant controlling risk positioning. However, the most popular measure of risk is the Value-at-risk (VaR) [4], where the objective function (1) becomes:

$$\underset{\boldsymbol{x}}{\text{Maximize}} \quad E[f(\boldsymbol{x}, \boldsymbol{Y})] - \alpha\text{-VaR}[f(\boldsymbol{x}, \boldsymbol{Y})] = \mu_F - \alpha\text{-VaR} \ , \tag{5}$$

however results reported in [5] and [6] show that it is difficult to work with when the random variables are not "normally" distributed, and efficient algorithms for optimization of VaR in high-dimensional settings are still not available. An additional risk metric similar to VaR, which has superior properties in many respects, is the Conditinal Value-at-Risk (CVaR). CVaR was first employed because the VaR does not provide an indication of the extent of losses that might be suffered beyond the amount indicated by this measure, i.e. the volatility of the losses remains unknown. Using this risk metric, the objective function (1) becomes:

$$\underset{\boldsymbol{x}}{\text{Maximize}} \quad E[f(\boldsymbol{x}, \boldsymbol{Y})] - \alpha\text{-CVaR}[f(\boldsymbol{x}, \boldsymbol{Y})] = \mu_F - \alpha\text{-CVaR} \ . \tag{6}$$

This alternative measure has gained widespread specially since Rockafellar and Uryasev [5, 7] proposed a new technique based on scenarios which calculates VaR and optimizes CVaR simultaneously.

The aim of this paper is to introduce a new approach to solve a certain class of stochastic programming problems defined by (1)-(3), which is inspired by structural engineering methods [see 8, 9, 10, 11, 12, 13, 14]. The

method relies on first-order reliability methods and mathematical programming decomposition techniques.

Note that the method proposed in this paper has the following advantages with respect to existing ones:

1. It can be used with any number of random variables, filling the gap of quantile optimization in high-dimensional settings.
2. The random variables need not be distributed in any particular manner.

However, the methodology is only suitable for solving stochastic programming problems where the random variables are modeled using the joint probability distribution function instead of scenarios or finite sampling.

The rest of the paper is organized as follows. Section 2 provides an introduction to first-order reliability methods, and describes the relationships between some of the existing risk metrics and the one proposed in the paper. In Section 3 a detailed description of the method for solving the stochastic programming problem is presented. In Section 4 results from a realistic case study pertaining to the self-scheduling of a power producer are described, analyzed, and compared with existing approaches. Finally, in Section 5 some relevant conclusions are duly drawn.

2. First-Order Reliability Based Methods (FORM)

In this section, a general framework for characterizing the cumulative distribution function of $f(\boldsymbol{x}, \boldsymbol{y})$ is introduced.

The proposed procedure to solve problem (1)-(3) involves two steps: i) decision making, and ii) quantile evaluation for given values of the decision variables. We advocate the use of first-order reliability methods for the second step since they can be applied to any joint probability distribution function of the random variables ($\boldsymbol{Y}$), making it appropriate for high dimensional settings.

For given values of the decision variables $\boldsymbol{x}$, the objective function of problem (1)-(3) involves several random variables $(Y_1, \ldots, Y_n)$, which makes $f(\boldsymbol{x}, \boldsymbol{y})$ to be also a random variable. Assuming the stochastic character of the objective function we could be interested on i) calculating the probability of this objective function being lower than or equal to a given threshold q_α (i.e. its cumulative distribution function (CDF)), or conversely, ii) what is the objective function value so that the probability of the random objective

function of being lower than or equal to this particular threshold is equal to a given value $1-\alpha$ (i.e. the quantile q_α).

The problem of getting the objective function CDF of a particular value q_α selected by the decision-maker can be achieved as follows. The random variables involved belong to an n-dimensional space, which can be divided into two regions with respect to the limit-state equation $f(\boldsymbol{x},\boldsymbol{y}) = q_\alpha$, the unacceptance and the acceptance regions. That is,

$$\begin{array}{l} U \equiv \{(y_1, y_2, \ldots, y_n)\} | f(\boldsymbol{x}, y_1, y_2, \ldots, y_n) > q_\alpha\}, \\ A \equiv \{(y_1, y_2, \ldots, y_n)\} | f(\boldsymbol{x}, y_1, y_2, \ldots, y_n) \leq q_\alpha\}, \end{array} \tag{7}$$

where the unacceptance region U corresponds to realizations whose f-values are greater than q_α and the acceptance region A to f-values lower than or equal to q_α.

The CDF of the objective function can be calculated using the joint probability distribution function of all random variables involved, by means of the following integral over the acceptance region A:

$$\mathrm{CDF}(q_\alpha; \boldsymbol{x}, \boldsymbol{\kappa}) = \int\limits_{f(\boldsymbol{x},\boldsymbol{y})-q_\alpha \leq 0} f_{\boldsymbol{Y}}(\boldsymbol{y}, \boldsymbol{\kappa}) dy_1 dy_2 \ldots dy_n, \tag{8}$$

where $\boldsymbol{\kappa}$ is the parameter vector of the model defining the random variability and dependence structure of the random variables involved (standard deviations, variation coefficients, correlations, covariance matrices, parameters of ARMA models, etc.).

Rather than using approximate (and numerical) methods to perform the integration required in (8), the non-normal dependent distributions can be transformed into equivalent normal distributions that can be integrated straightforwardly. Using the appropriate transformations, (8) can be evaluated by solving the following optimization problem:

$$\beta = \underset{\boldsymbol{y}}{\text{Minimize}} \quad \sqrt{\sum_{j=1}^{n} z_j^2}\ , \tag{9}$$

subject to

$$f(\boldsymbol{x},\boldsymbol{y}) - q_\alpha = 0 \tag{10}$$

$$\boldsymbol{T}(\boldsymbol{y},\boldsymbol{\kappa}) = \boldsymbol{z}, \tag{11}$$

where (10) is the strict acceptance condition, and (11) is the transformation [15, 16] that converts $\boldsymbol{y}$ into a standard set of independent random normal variables $\boldsymbol{z}$.

The optimal objective function solution of (9) is called the *reliability* index (β), which constitutes the risk metric proposed, whereas the optimal vectors $\boldsymbol{y}^{\mathrm{ML}}$ and $\boldsymbol{z}^{\mathrm{ML}}$ are *points of maximum likelihood* in the initial and transformed space, respectively, which are related by (11). Note that the solution of problem (9)-(11) corresponds to the point, i.e. particular instances of the random variables, inside the acceptance region whose value of the joint probability distribution function is maximum, representing the most likely values of the random variables that produce the outcome $f(\boldsymbol{x}, \boldsymbol{y}) = q_\alpha$.

The CDF of the objective function is related to the reliability index by the relation

$$\mathrm{CDF}(q_\alpha) = \Phi(-\beta), \tag{12}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variables. This method provides the exact CDF if the limit-state equation is linear in the standard normal random space, i.e. if the resulting objective function distribution is normally distributed $f(\boldsymbol{x}, \boldsymbol{y}) \sim N(\mu_F, \sigma_F^2)$; otherwise, it is an approximate method.

From expression (10) particularized for the optimal solution of problem (9)-(11) it holds:

$$q_\alpha = f(\boldsymbol{x}, \boldsymbol{y}^{\mathrm{ML}}), \tag{13}$$

which implies that assuming certain values of the decision variables $\boldsymbol{x}$ there is a unique correspondence between the cumulative distribution function of the objective function for a given value q_α and the point of maximum likelihood $\boldsymbol{y}^{\mathrm{ML}}$. This relationship is exploited afterwards to solve problem (1)-(3).

In Figure 1, a 2-D graphical interpretation of the FORM method is shown. In the standard normal random space $\boldsymbol{z}$, the joint probability distribution function contours are concentric circumferences at the origin such that the further the point (z_1, z_2) is away from the origin, the lower the value of the joint probability distribution function. The solution of problem (9)-(11) provides the closest point to the origin inside the acceptance region. The limit-state equation is then linearized at the design point $\boldsymbol{z}^{\mathrm{ML}}$.

A key step for the successful application of this technique is the transformation (11). In [15, 16, 17, 18] and [13] several methods for transforming arbitrary and dependent random variables into independent standard normal

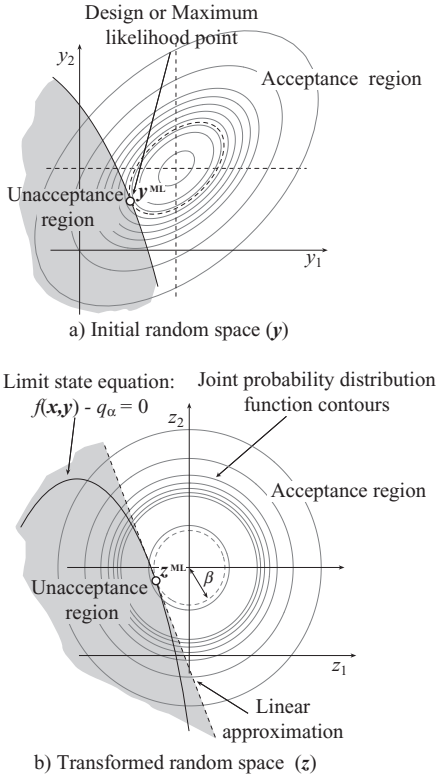

Figure 1: Joint probability distribution function contours, limit-state equation, and design points in: a) the initial random space $\boldsymbol{y}$, and b) the unit standard normal random space $\boldsymbol{z}$.

variables are presented. Note that the selection of the appropriate transformation is strongly dependent on the information available.

2.1. Relationship Among Risk Metrics

First-order reliability methods rely on the calculation of the reliability index β as the corresponding risk metric, the relationships between some of the most common risk metrics and this index are provided. Note that for solving the stochastic programming problem (1)-(3) the quantile (q_α) to be maximized must be selected through the confidence level α, i.e. $1 - \alpha = \text{CDF}(q_\alpha)$. From (12) it yields:

$$1 - \alpha = \Phi(-\beta) \Rightarrow \beta = -\Phi^{-1}(1 - \alpha), \tag{14}$$

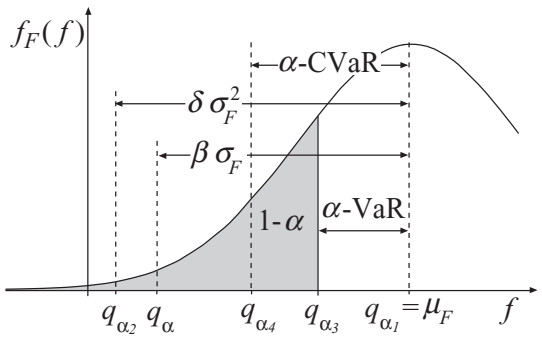

Figure 2: Probability distribution function of the objective function random variable for given values of the decision variables.

which represents the reliability index associated with the confidence level α. Note that we consider the reliability index positive for α-values greater than 0.5.

Optimization methods under uncertainty using existing risk metrics optimize also a given quantile q_α of the objective function distribution (see Figure 2). This quantile corresponds to:

1. In the **expected value** analysis, q_{α_1} corresponds to a central value (mean or close to the mean) of the objective function μ_F ($q_{\alpha_1} = \mu_F$), so that the optimal values of the decision variables ($\boldsymbol{x}^*$) ensure the greatest possible central value for the objective function distribution.
2. In the **mean-variance** analysis, q_{α_2} corresponds to a value of the objective function equal to $\mu_F - \delta\sigma_F^2$, so that the optimal values of the decision variables ($\boldsymbol{x}^*$) ensure the greatest possible value of $q_{\alpha_2} = \mu_F - \delta\sigma_F^2$ for a given value of parameter δ.
3. For the **Value-at-Risk** analysis, q_{α_3} corresponds to the value of the objective function equal to $\mu_F - (\alpha\text{-VaR})$, so that $\text{Prob}(f(\boldsymbol{x}, \boldsymbol{y}) \leq q_{\alpha_3}) = 1 - \alpha$. In this case, the optimal values of the decision variables ($\boldsymbol{x}^*$) ensure the maximum possible value of $q_{\alpha_3} = \mu_F - (\alpha\text{-VaR})$ while holding the probability constraint. Note that the integral of the probability distribution function from $-\infty$ to q_{α_3} (light gray shadow in Figure 2) corresponds to the desired probability as a function of the confidence level $(1 - \alpha)$.
4. For the **CVaR** analysis, q_{α_4} corresponds to the value of the objective function equal to $\mu_F - (\alpha\text{-CVaR})$, i.e. the mean value of the random

variable $f(\boldsymbol{x}, \boldsymbol{y})$ conditioned by $f(\boldsymbol{x}, \boldsymbol{y}) \leq \mu_F - (\alpha\text{-VaR})$. The optimal values of the decision variables ($\boldsymbol{x}^*$) also ensure the greatest possible value for $q_{\alpha_4} = \mu_F - (\alpha\text{-CVaR})$.

For the different risk metrics to be equivalents, the quantiles associated with those risk metrics must coincide:

1. For the expected value analysis $q_\alpha = q_{\alpha_1} = \mu_F$, using (14) results in a confidence level of $\alpha = 0.5$ and $\beta = 0$.
2. The equivalence between the mean-variance and the reliability index formulations comes from the expression ($q_\alpha = q_{\alpha_2}$) $\mu_F - \delta\sigma_F^2 = \mu_F - \beta\sigma_F$, and $\delta\sigma_F = \beta = -\Phi^{-1}(1-\alpha)$.
3. For the VaR method, the relationship is $\mu_F - \alpha\text{-VaR} = \mu_F - \beta\sigma_F$ ($q_\alpha = q_{\alpha_3}$), thus,

$$\alpha\text{-VaR} = \beta\sigma_F = -\Phi^{-1}(1-\alpha)\,\sigma_F, \tag{15}$$

   which coincides with (14).
4. For the CVaR, the analysis is more involved. Considering that $z = (q_{\alpha_4} - \mu_F)/\sigma_F$ and using Bayes theorem, q_{α_4} for a given confidence level α is:

$$\begin{aligned} q_{\alpha_4} &= \mu_F + \frac{\sigma_F}{1-\alpha}\int_{-\infty}^{-\beta} z f_Z(z)dz = \\ &= \mu_F - \sigma_F \frac{1}{1-\alpha}\frac{\exp\left(-\beta^2/2\right)}{\sqrt{2\pi}}, \end{aligned} \tag{16}$$

   which is valid for $\alpha \geq 0.5$. Thus,

$$\alpha\text{-CVaR} = \frac{1}{1-\alpha}\frac{\exp\left(-\beta^2/2\right)}{\sqrt{2\pi}}\sigma_F. \tag{17}$$

   Note that (15) and (17) provide the relationships between α-VaR and α-CVaR with the reliability index β.
   At this point, the reliability index β can be defined as the number of standard deviations by which the mean value of the objective function μ_F exceeds q_{α_4}.
   By combining (14), (15), (17) and simplifying, the following relationship is obtained:

$$\alpha^* = 1 - \Phi\left(-\frac{\exp\left\{-(-\Phi^{-1}(1-\alpha))^2/2\right\}}{(1-\alpha)\sqrt{2\pi}}\right);\ \forall\alpha \geq 0.5, \tag{18}$$

which provides the relationship between confidence levels α^* and α for the α^*-VaR and α-CVaR approaches to be equivalent.

Using the expressions derived above, it is possible to obtain the approximate relationships between the different risk metrics to obtain a similar statistical distribution of the objective function for given values of the decision variables $\boldsymbol{x}$. These relations are exact if the resulting objective function distribution is normally distributed.

3. Method for solving the stochastic programming problem

Problem (1)-(3) maximizes a given quantile q_α for an α-value decided by the decision maker. From (14) the desired reliability index is also known. It is also known from (13) that for given values of the decision variables $\boldsymbol{x}$ there is a unique point of maximum likelihood $\boldsymbol{y}^{\mathrm{ML}}$ which provides the desired quantile, i.e. $q_\alpha = f(\boldsymbol{x}, \boldsymbol{y}^{\mathrm{ML}})$.

The goal of problem (1)-(3) is to select the values of the decision variables $\boldsymbol{x}$ in order to maximize the corresponding quantile q_α, which implies holding the following conditions:

$$q_\alpha = \underset{\boldsymbol{x}}{\text{Maximum}} \quad f(\boldsymbol{x}, \boldsymbol{y}^{\mathrm{ML}}) \ , \tag{19}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{20}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \tag{21}$$

whose optimal solution is $\boldsymbol{x}^*$, and so that $\boldsymbol{y}^{\mathrm{ML}}$ truly corresponds to q_α, i.e.

$$\beta = -\Phi^{-1}(1-\alpha) = \underset{\boldsymbol{y}}{\text{Minimimum}} \quad \sqrt{\sum_{j=1}^{n} z_j^2} \ , \tag{22}$$

subject to

$$f(\boldsymbol{x}^*, \boldsymbol{y}) = q_\alpha \tag{23}$$

$$\boldsymbol{T}(\boldsymbol{y}, \boldsymbol{\kappa}) = \boldsymbol{z}, \tag{24}$$

whose optimal solution is the point of maximum likelihood $\boldsymbol{y}^{\mathrm{ML}}$ used in (19). Note that the optimal objective function q_α and variables $\boldsymbol{x}^*$ from problem

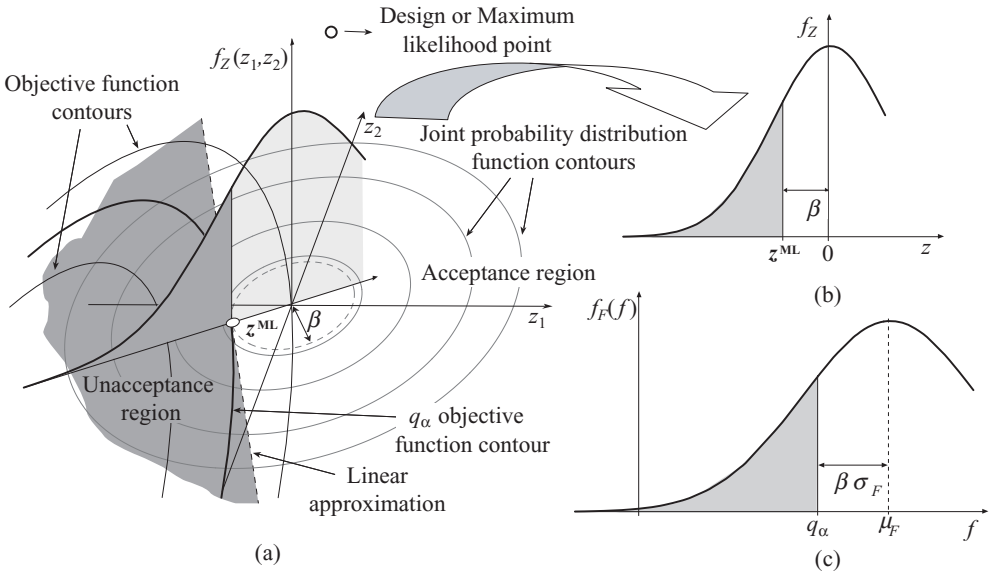

Figure 3: One and bi-dimensional graphical interpretation of the FORM method using for optimal risk management.

(19)-(21) are used in (23), and conversely, the optimal variables $\boldsymbol{y}^{\mathrm{ML}}$ from problem (22)-(24) are used in (19).

In Figure 3 (a), a 3-D graphical interpretation of the risk management method using FORM is shown. Note that the objective function q_α contour in the independent standard normal random space is tangent to the circumference centered at the origin with a radius equal to $\beta = -\Phi^{-1}(1-\alpha)$. Using the linear approximation of the contour (limit-state equation) at the point of maximum likelihood, the problem reduces to one dimension and the objective function is normally distributed $(F-\mu_F)/\sigma_F \sim N(0,1^2)$ (see Figure 3 (b)), i.e. $F \sim N(\mu_F, \sigma_F^2)$ (see Figure 3 (c)).

Unfortunately, problem (1)-(3) cannot be solved directly because it involves two nested but decoupled optimization formulations: decision making *per se*, and quantile evaluation. The procedure presented in this paper solves iteratively two different problems until the quantile associated with the decision variables coincides with the target quantile.

To make the problem tractable we use the following lemma:

Lemma 3.1. *The problem (22)-(24) at the optimum is equivalent to the fol-*

lowing optimization problem [see 19]:

$$q_\alpha = \underset{\boldsymbol{y}}{Minimum} \quad f(\boldsymbol{x}^*, \boldsymbol{y}) \ , \tag{25}$$

subject to

$$\sqrt{\sum_{j=1}^{n} z_j^2} \quad = \quad \beta = -\Phi^{-1}(1-\alpha) \tag{26}$$

$$\boldsymbol{T}(\boldsymbol{y}, \boldsymbol{\kappa}) \quad = \quad \boldsymbol{z}. \tag{27}$$

Proof of this lemma is provided in Appendix Appendix A. Problem (25)-(27) allows us for given values of the decision variables to update the point of maximum likelihood which fulfills the confidence level criterion (14).

The proposed iterative scheme solves problem (1)-(3) based on decomposition techniques [see 20, 14] and it is described step by step on the following algorithm.

Algorithm 3.1. (Iterative method).

Input: Target confidence level α, objective function, constraints, the statistical description of the involved random variables $\boldsymbol{Y}$ and the tolerance of the process ε. These data are selected by the decision maker.

Step 1: Initialization. Initialize the iteration counter to $\nu = 1$, and the point of maximum likelihood to the mean values of the random variables $\boldsymbol{y}_{(\nu)}^{\mathrm{ML}} = \bar{\boldsymbol{y}}$. We select this initial point since it corresponds to the quantile associated with $\alpha = 0.5$. Alternative selections are also valid.

Step 2: Solving the decision variable problem at iteration ν. Assuming that the actual values of the point of maximum likelihood correspond to the the desired quantile, obtain new values of the decision variables for the actual iteration by solving problem (19)-(21). The result provides values of the decision variables, $\boldsymbol{x}_{(\nu)}$, which satisfy the feasibility constraints (20)-(21).

Note that at the first iteration ($\nu = 1$) this problem corresponds to the expected value analysis.

Step 3: Convergence checking. If the iteration counter is equal to 1 ($\nu = 1$) go to *Step 4*, otherwise, proceed to check convergence, i.e., if $||\boldsymbol{x}_{(\nu)} - \boldsymbol{x}_{(\nu-1)}|| \leq \varepsilon$ go to *Step 5*, else continue in *Step 4*.

Step 4: Evaluating the new point of maximum likelihood. The actual value of the decision variables $\boldsymbol{x}_{(\nu)}$ is the solution of the global problem (19)-(24) if and only if $\boldsymbol{y}^{\mathrm{ML}}_{(\nu)}$ truly corresponds to the desired quantile q_α, which generally does not hold.

Thus, we update the point of maximum likelihood for the next iteration $\boldsymbol{y}^{\mathrm{ML}}_{(\nu+1)}$ forcing it to hold the quantile constraint. This is achieved solving problem (25)-(27) for the actual values of the decision variables $\boldsymbol{x}_{(\nu)}$. The iteration counter must then be updated $\nu \rightarrow \nu + 1$ before proceeding to *Step 2*.

Step 5: Output. The solution for a given tolerance corresponds to $\boldsymbol{x}^* = \boldsymbol{x}_{(\nu)}$, $\boldsymbol{y}^{\mathrm{ML}} = \boldsymbol{y}^{\mathrm{ML}}_{(\nu)}$ and $q^*_F = f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}})$.

■

Extensive numerical simulations involving different problems show that the process converges quickly to the optimal solution. Note that by using decomposition techniques, both problems –optimal management (decision making) and risk (quantile evaluation)– are treated separately.

The convergence characteristics of this iterative method are discussed in Appendix Appendix B.

4. Case Study: Self-Scheduling Problem

We consider below the self-scheduling problem of a power producer. Electric energy can be traded in two markets, a pool and a futures market. The pool consists of a day-ahead market while the futures market allows trading electricity up to one year ahead. The futures market presents a lower average price than the pool but involves reduced volatility. Thus, it allows hedging against the financial risk inherent in pool price volatility [see 21].

A power producer needs to define its involvement in both the pool and the futures markets so that its profit is maximized within a particular risk level based on profit volatility. The producer decides how much power to sell in the futures market at a fixed price spanning the time horizon, and for each period of the time horizon it decides how much power to sell in the pool. The considered profit-maximization problem can be formulated as follows:

$$\underset{p^{\mathrm{P}}_1, p^{\mathrm{P}}_2, \ldots, p^{\mathrm{P}}_T, p^{\mathrm{C}}}{\text{Maximize}} \quad q\left(\sum_{t=1}^{T}\left(\lambda_t p^{\mathrm{P}}_t - c_t\right) + \lambda^{\mathrm{C}} p^{\mathrm{C}} N_T\right), \tag{28}$$

subject to

$$(p_1^{\mathrm{P}}, p_2^{\mathrm{P}}, \ldots, p_T^{\mathrm{P}}, p^{\mathrm{C}}) \quad \in \quad \Pi, \tag{29}$$

where $q(\cdot)$ represents a quantile of the random objective function, p_t^{P} is power sold in the pool during period t (optimization variable), p^{C} is power sold in the futures market (optimization variable), λ_t is the pool price in period t, λ^{C} is the futures market price, c_t is the production cost during period t, and N_T is the number of time periods considered. Constraint (29) represents operation constraints. Further information on the definition of these functions can be found in [3].

Data for the power producer are taken from [3]. The futures market price is taken to be 36 \$/MWh. Pool prices are random variables with mean $\boldsymbol{\lambda}^{\mathrm{est}}$ and the covariance matrix $\boldsymbol{V}_\lambda$ taken from [3].

The problem is analyzed under different statistical assumptions: the case that pool prices are assumed to be normally distributed, and the case these are log-normally distributed and the costs are also random variables. Note that all problems are solved using Package GAMS (http://www.gams.com) on a Linux-based server with one processor clocking at 2.6 GHz and 32 GB of RAM.

The aim of the producer is maximize the profit that has a probability of being exceeded equal to the confidence level α. Thus, only parameter α has to be selected in order to obtain an appropriate trade-off between profit and risk.

4.1. Case 1: Gaussian distributed pool prices

We have chosen this set up for comparison purposes. Since pool prices are assumed to be normally distributed, once power production in pool market $\boldsymbol{p}^{\mathrm{P}}$, power production in futures market p^{C} and costs $\boldsymbol{c}$ are known, the profit distribution function is normally distributed with the following parameters:

$$\mu_P = \sum_{t=1}^{T} \left(\lambda_t^{\mathrm{est}} p_t^{\mathrm{P}} - c_t\right) + \lambda^{\mathrm{C}} p^{\mathrm{C}} N_T, \text{ and } \sigma_P^2 = (\boldsymbol{p}^{\mathrm{P}})^T \boldsymbol{V}_\lambda \boldsymbol{p}^{\mathrm{P}}, \tag{30}$$

which allows solving the different risk models using the quantile expressions derived in subsection 2.1. Note that the proposed iterative method provides exact results for this particular case since profit is normally distributed.

In order to compare the above procedures, the following models have been solved:

1. VaR and CVaR approaches using a confidence level $\alpha = 0.95$ and quantile expressions derived in subsection 2.1, which are exact. These problems are mixed-integer nonlinear programming problems which are solved using solver SBB [22].
2. VaR and CVaR approaches using a confidence level $\alpha = 0.95$ and the proposed iterative algorithm, which is also exact since the resulting profit distribution is normal. For this case, the decision making problem (19)-(21) is also a mixed-integer nonlinear programming problems which is solved using SBB, and the quantile updating problem (25)-(27) is a nonlinear programming problem which is solved using MINOS [23].
3. CVaR approach using the same confidence level $\alpha = 0.95$ but based on different number (10, 50, 100, 150, 200 and 300) of synthetically generated scenarios from the price distribution. This problem is also a mixed-integer nonlinear programming problem.

Table 1 provides the objective function optimal value (OFV), the mean (μ_P), the standard deviation (σ_P), the variation coefficient of the profit distribution (ρ_P), the number of equations (n_{eq}), the number of continuos variables (n_{cv}), the number of discrete variables (n_{dv}) of the corresponding model, and the cpu time required to attain the optimal solution. Considering these results, the following observations are pertinent:

1. VaR using both methods provide the same solution within a similar computational time (see first two rows in Table 1).
2. Analogously, both continuos CVaR approaches attain the same solution, which is more conservative than using VaR, yielding lower mean and standard deviation values (third and fourth rows in Table 1).
3. In the results given by the CVaR approach using scenarios, the solution oscillates for different number of scenarios. Note that this result is not surprising since CVaR solution is based only on the benefits corresponding to the worst 5% scenarios, i.e. 1, 2, 5, 7, 10 and 15 scenarios, respectively, and analogously to the Monte Carlo simulation a minimum number of sampled scenarios is necessary to estabilice the solution.
4. Computational time for the scenario approach increases exponentially due to the size of the problem to be solved, which makes the calculation of this minimum threshold computationally very expensive.

	OFV [$]	μ_P [$]	σ_P [$]	ρ_P	n_{eq}	n_{cv}	n_{dv}	cpu time [s]
α-VAR	29471.29	30741.30	772.11	0.0251	289	221	96	2.11
α-VAR$_{\text{it}}$	29471.29	30741.21	772.06	0.0251	290	221	96	8.43
α-CVAR	29171.91	30541.50	663.97	0.0217	289	221	96	1.47
α-CVAR$_{\text{it}}$	29171.91	30541.35	663.90	0.0217	290	221	96	13.92
$n = 10$	30797.32	31964.82	820.77	0.0257	1821	1115	96	17.09
$n = 50$	29417.89	30876.77	877.86	0.0284	8621	5075	96	361.27
$n = 100$	29373.38	31040.94	908.71	0.0293	17121	10025	96	1487.79
$n = 150$	29583.63	31135.76	847.08	0.0272	25621	14975	96	3843.82
$n = 200$	29405.58	30939.21	773.91	0.0250	34121	19925	96	6633.87
$n = 300$	29459.89	31048.80	837.36	0.0270	61121	29825	96	19916.61

Table 1: Results of the case study using different risk models for normally distributed pool prices.

5. Note also that for these particular scenarios the solution overestimates the CVaR. However, in order to obtain statistically sound conclusions the expected solution value for different scenarios should be calculated. This is a cumbersome process which makes the proposed method more appropriate for these particular case.

It is worth mentioning that results for the VaR and CVaR approaches with gaussian distributed pool prices are obtained after six and ten iterations of the proposed method.

4.2. Case 2: Log-normally distributed pool prices and logistic distributed production costs

To show the applicability of the proposed method in high dimensional settings using non-normal random variables, log-normally distributed pool prices are considered. In addition, production costs c_t are assumed to be independent random variables following the logistic distribution. This hypothesis attempts to account for the possible influence of costs uncertainty in the self-scheduling problem and it has been selected for illustration purposed, i.e. it is not based on data fits.

At this point, it is important to select the right transformation of the random variables. In this case, since $\ln \boldsymbol{\lambda}$ are normally distributed random

variables, the orthogonal transformation is adequate [see 13]:

$$\boldsymbol{Z}_{\boldsymbol{\lambda}} = \boldsymbol{A}(\ln \boldsymbol{\lambda} - \boldsymbol{\mu}_{\ln \lambda}),$$

where $\boldsymbol{A}$ is the inverse of the Cholesky decomposition of the variance-covariance matrix $\boldsymbol{V}_{\ln \lambda}$, and $\boldsymbol{\mu}_{\ln \lambda}$ is the mean values vector of the logarithm of the prices whose components can be obtained as follows:

$$\mu_{\ln \lambda_i} = \ln \frac{\mu_{\lambda_i}}{\sqrt{1 + \left(\dfrac{\sigma_{\lambda_i}}{\mu_{\lambda_i}}\right)^2}}.$$

The components of the covariance matrix $\boldsymbol{V}_{\ln \lambda}$ are [18]:

$$V_{\ln \lambda_{ij}} = \sigma_{\ln \lambda_i} \rho'_{ij} \sigma_{\ln \lambda_j}, \qquad \rho'_{ij} = \frac{\ln \left(1 + \dfrac{\rho_{ij} \sigma_{\lambda_i} \sigma_{\lambda_j}}{\mu_{\lambda_i} \mu_{\lambda_j}}\right)}{\sigma_{\ln \lambda_i} \sigma_{\ln \lambda_j}},$$

where $\sigma_{\ln \lambda_i} = \sqrt{\ln \left(1 + \left(\sigma_{\lambda_i} / \mu_{\lambda_i}\right)^2\right)}$, ρ_{ij} is the element (i, j) of the correlation matrix in the original space and ρ'_{ij} is the element (i, j) of the correlation matrix in the logarithmic space.

For the costs, the required transformation is [15]:

$$\Phi(z_{c_t}) = F_C(c_t) = \frac{1}{1 + \exp\left(-\pi(c_t - \bar{c})/(\sqrt{3}\bar{c}\delta)\right)}; \ \forall t, \tag{31}$$

where z_{c_t} is the associated transformed variable, F_C is the cumulative distribution function of the costs, $\bar{c}$ is the mean cost which is considered equal for every time period, and δ is the coefficient of variation which is taken equal to 0.05 (5%).

Note that in Case 1 both VaR and CVaR models could be solved either by using the quantile expressions derived in Section 2.1 or the iterative method provided in Section 3. However, for this particular case, only the following models can be solved:

1. VaR and CVaR approaches using a confidence level $\alpha = 0.95$ and the proposed iterative algorithm.
2. CVaR approach using the same confidence level $\alpha = 0.95$ but based on different number (10, 50, 100, 150, 200 and 300) of synthetically generated scenarios from the price and costs distributions.

	OFV [$]	μ_P [$]	σ_P [$]	ρ_P	n_{eq}	n_{cv}	n_{dv}	cpu time [s]
α-VAR$_{\text{it}}$	28561.03	30902.40	1423.45	0.04606	290	221	96	10.63
α-CVAR$_{\text{it}}$	27976.46	30838.05	1387.30	0.0450	290	221	96	13.21
$n = 10$	29639.55	31323.91	1483.45	0.0474	1821	1115	96	20.07
$n = 50$	28055.41	31091.36	1681.44	0.0541	8621	5075	96	467.92
$n = 100$	28182.63	31089.55	1452.33	0.0467	17121	10025	96	2050.89
$n = 150$	28340.59	31123.89	1427.951	0.0459	25621	14975	96	4221.47
$n = 200$	27936.49	30909.86	1520.84	0.0492	34121	19925	96	7882.51
$n = 300$	28079.63	31124.15	1465.23	0.0471	51121	29825	96	29491.02

Table 2: Results of the case study using different risk models for lognormally distributed pool prices and logistic distributed costs.

Table 2 provides analogous results as those in Table 1. Considering these results, the following observations are pertinent:

1. The proposed method solves both VaR and CVaR approaches within a reasonable amount of cpu time.
2. CVaR solution is also more conservative than the one using VaR, yielding lower mean and standard deviation values.
3. Results given by the CVaR approach using scenarios oscillate for different number of scenarios and computational time increases exponentially with the number of scenarios.
4. Both methods provide similar solutions for large number of scenarios, but the latter is computationally more involved and it can not to be used to maximize VaR.

The results for the VaR and CVaR approaches with log-normally distributed pool prices and logistic distributed costs are obtained after seven and nine iterations of the proposed algorithm.

5. Conclusions

Based on first-order reliability methods and decomposition techniques, this paper proposes an iterative method for solving a certain type of stochastic programming problems, which constitute an essential part of many different decision-making processes. It can also be used for optimizing quantiles

in any context outside finance. The method is specially suitable for problems presenting the following characteristics:

1. The joint probability distribution of the random variables is given or can be estimated parametrically.
2. Their distribution do not depend on the decision variables.
3. The random variables only affect the objective function.

Comprehensive simulations carried out for different case studies show the effectiveness of the proposed methodology in high-dimensional settings dealing with non-normal random variables. It constitutes an effective alternative to scenario based methods, which face problems such as i) constructing appropriate scenarios from the probability distributions to approximate uncertainty, and ii) size, which can lead to problem intractability.

Additional advantages of the proposed method are:

1. Relationships between the reliability index and alternative risk metrics are given.
2. The decision variable and quantile evaluation problems are decoupled, allowing a richer interpretation of the solution.
3. The decomposition procedure presents good convergence properties and computational behavior, making it suitable to solve large scale problems.

The utilization of first-order reliability methods and decomposition techniques brings new possibilities for solving different stochastic programming problems, such as those including random variables affecting constraints. This constitutes a subject for further research.

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Appendix A. Proof of Lemma 3.1

The first-order Karush-Kuhn-Tucker (KKT) optimality conditions for the nonlinear programming problem (NLPP) (22)-(24) at the optimal solution $(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{z}^{\mathrm{ML}})$ are:

$$\nabla\left(\sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}}\right) + \lambda_y \nabla\left(f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}) - q_F\right) + \sum_{j=1}^{n} \mu_{y_j} \nabla\left(T_j(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) - z_j^{\mathrm{ML}}\right) = \mathbf{0} \quad \text{(A.1)}$$

$$f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}) = q_F \quad \text{(A.2)}$$

$$T_j(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) = z_j^{\mathrm{ML}};\ j = 1, \ldots, n \quad \text{(A.3)}$$

where λ_y and $\boldsymbol{\mu}_y$ are the dual variables associated with constraints (23)-(24), respectively. If i) $\lambda_y \neq 0$, ii) no redundant constraints exist, and iii) an optimal solution exists, then we can rewrite (A.1)-(A.3) as:

$$\frac{1}{\lambda_y}\nabla\left(\sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}} - \beta^*\right) + \nabla\left(f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}) - q_F\right) + \sum_{j=1}^{n} \frac{\mu_{y_j}}{\lambda_y} \nabla\left(T_j(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) - z_j^{\mathrm{ML}}\right) = \mathbf{0} \quad \text{(A.4)}$$

$$\sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}} = \beta^* \quad \text{(A.5)}$$

$$T_j(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) = z_j^{\mathrm{ML}};\ j = 1, \ldots, n \quad \text{(A.6)}$$

which are the first-order KKT optimality conditions for the NLPP (25)-(27) at the optimal solution.

Appendix B. Iterative FORM Convergence Analysis

Considering that i) problem (19)-(24) is feasible, ii) the functions involved are twice continuously differentiable, and iii) the optimal solution is $(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{z}^{\mathrm{ML}}, \boldsymbol{\lambda}_x^*, \lambda_{yz}^*, \boldsymbol{\mu}_{yz}^*)$, the first-order KKT optimality conditions for the decomposed problem (19)-(21) at the optimal solution are:

$$\nabla_x \mathcal{L}_x(\boldsymbol{x}^*, \boldsymbol{\lambda}_x^*) = \nabla_x f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}) + \boldsymbol{\lambda}_x^{*^T} \nabla_x \boldsymbol{h}^*(\boldsymbol{x}^*) = 0 \quad \text{(B.1)}$$

$$\boldsymbol{h}^*(\boldsymbol{x}^*) = \mathbf{0}, \quad \text{(B.2)}$$

where $\boldsymbol{h}^*(\boldsymbol{x}^*)$ includes the equality constraints (20) and the active inequality constraints (21). The KKT optimality conditions for problem (22)-(24) are:

$$\nabla_{yz} \mathcal{L}_{yz}(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{z}^{\mathrm{ML}}, \lambda_{yz}^*, \boldsymbol{\mu}_{yz}^*) = \nabla_{yz} \left(f(\boldsymbol{x}^*, \boldsymbol{y}^{\mathrm{ML}}) - q_F \right)$$

$$+\lambda_{yz}^* \nabla_{yz} \left(\sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}} \right) + \boldsymbol{\mu}_{yz}^{*^T} \nabla_{yz} \left(\boldsymbol{T}(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) - z_j^{\mathrm{ML}} \right) = \mathbf{0} \quad \text{(B.3)}$$

$$\sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}} = \beta^* \quad \text{(B.4)}$$

$$\boldsymbol{T}(\boldsymbol{y}^{\mathrm{ML}}, \boldsymbol{\kappa}) = \boldsymbol{z}^{\mathrm{ML}}. \quad \text{(B.5)}$$

In the proposed iterative FORM, the Newton-Raphson method is used to solve these two system of equations, establishing that the first-order conditions make the gradient equal to zero. Starting from initial values $\boldsymbol{x}_{(k)}$, $\boldsymbol{y}_{(k)}$, $\boldsymbol{z}_{(k)}$, $\boldsymbol{\lambda}_x^{(k)}$, $\lambda_{yz}^{(k)}$ and $\boldsymbol{\mu}_{yz}^{(k)}$ sufficiently close to the optimal solution, the search directions are obtained solving the following systems of equations:

$$\begin{bmatrix} \nabla_{xx} \mathcal{L}_x^{(k)} & \nabla_x \boldsymbol{h}_{(k)}^{*^T} \\ \nabla_x \boldsymbol{h}_{(k)}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{x}^{(k)} \\ \Delta \boldsymbol{\lambda}_x^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla_x \mathcal{L}_x^{(k)} \\ \boldsymbol{h}^*(\boldsymbol{x}_{(k)}, \boldsymbol{y}_{(k)}) \end{bmatrix} \quad \text{(B.6)}$$

$$[\mathrm{KKT}_a] [\Delta_a] = - [\nabla \mathcal{L}_a]$$

and

$$\begin{bmatrix} \nabla_{yzyz} \mathcal{L}_{yz}^{(k)} & \nabla_{yz} \beta_{(k)} & \nabla_{yz} \boldsymbol{T}_{(k)}^T \\ \nabla_{yz} \beta_{(k)} & 0 & \mathbf{0} \\ \nabla_{yz} \boldsymbol{T}_{(k)} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{y}^{(k)} \\ \Delta \boldsymbol{z}^{(k)} \\ \Delta \lambda_{yz}^{(k)} \\ \Delta \boldsymbol{\mu}_{yz}^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla_{yz} \mathcal{L}_{yz}^{(k)} \\ \beta = \sqrt{\sum_{j=1}^{n} z_j^{\mathrm{ML}^2}} - \beta^* \\ \boldsymbol{T}(\boldsymbol{y}_{(k)}, \boldsymbol{\kappa}) - \boldsymbol{z}_{(k)} \end{bmatrix} \quad \text{(B.7)}$$

$$[\mathrm{KKT}_b] [\Delta_b] = - [\nabla \mathcal{L}_b]$$

which at every iteration are solved in a distributed fashion. Note that, hereafter, letters a and b refer to problems (B.1)-(B.2) and (B.3)-(B.5), respectively. Thus, the descent directions for problems a and b, (Δ_a, Δ_b), in the

proposed decomposition algorithm can be obtained by solving the decomposable and approximate linear system of equations

$$\mathrm{KKT} \equiv \begin{pmatrix} \mathrm{KKT}_a & 0 \\ 0 & \mathrm{KKT}_b \end{pmatrix} \begin{pmatrix} \Delta_a \\ \Delta_b \end{pmatrix} = - \begin{pmatrix} \nabla \mathcal{L}_a \\ \nabla \mathcal{L}_b \end{pmatrix}. \tag{B.8}$$

However, for solving equations (B.1)-(B.5) using the Newton-Raphson method, the search directions for sub-problems a and b, (Δ_a, Δ_b), are computed by solving in each iteration a system of linear equations of the form

$$\mathrm{KKT}' \equiv \begin{pmatrix} \mathrm{KKT}'_a & \mathrm{KKT}'_{ba} \\ \mathrm{KKT}'_{ab} & \mathrm{KKT}'_b \end{pmatrix} \begin{pmatrix} \Delta_a \\ \Delta_b \end{pmatrix} = - \begin{pmatrix} \nabla_a \mathcal{L}' \\ \nabla_b \mathcal{L}' \end{pmatrix}, \tag{B.9}$$

where KKT'_a, KKT'_b, KKT'_{ab} and KKT'_{ba} are the Newton matrices [see 24] for problems a and b, defined as

$$\mathrm{KKT}'_a = \begin{bmatrix} \nabla_{xx}(\mathcal{L}_x^{(k)} + \mathcal{L}_{yz}^{(k)}) & \nabla_x \boldsymbol{h}_{(k)}^{*^T} \\ \nabla_x \boldsymbol{h}_{(k)}^{*} & \mathbf{0} \end{bmatrix},$$

$$\mathrm{KKT}'_b = \begin{bmatrix} \nabla_{yzyz}(\mathcal{L}_x^{(k)} + \mathcal{L}_{yz}^{(k)}) & \nabla_{yz}\beta_{(k)} & \nabla_{yz}\boldsymbol{T}_{(k)}^{T} \\ \nabla_{yz}\beta_{(k)} & 0 & \mathbf{0} \\ \nabla_{yz}\boldsymbol{T}_{(k)} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathrm{KKT}'_{ab} = \begin{bmatrix} \nabla_{xyz}(\mathcal{L}_x^{(k)} + \mathcal{L}_{yz}^{(k)}) & 0 & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \end{bmatrix}, \ \mathrm{KKT}'_{ba} = \mathrm{KKT}'^{T}_{ab}.$$

From these definitions, if at the optimal solution of problem (19)-(24) it holds that

$$\rho(\boldsymbol{I} - \mathrm{KKT}^{-1}\mathrm{KKT}') < 1, \tag{B.10}$$

then the proposed decomposition algorithm converges locally to the solution at a linear rate. Here $\rho(A)$ denotes the spectral radius of matrix A, while matrix $\boldsymbol{I}$ is the identity matrix. Condition (B.10) is related to the many results reported in the technical literature for the distributed solution of linear systems of equations, see for example [25] and [26]. Finally, note that by using the Newton method, the local rate of convergence for a centralized approach can be quadratic.

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