Optimal Cost Design with Sensitivity Analysis using Decomposition Techniques. Application to Composite Breakwaters

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Abstract

Minimizing the expected total cost of a structure, including maintenance and construction, is a difficult problem because of the presence in the objective function of the yearly failure rates, which have to be calculated by an optimization problem per each failure mode (FORM). In this paper a new method for the design of structures that minimizes the total expected costs of the structure during its lifetime based on Benders' decomposition is presented. In addition, some tools for sensitivity analysis are introduced, which make it possible to determine how the cost and yearly failure rates of the optimal solution are affected by small changes in the input data values. The proposed method is illustrated by its application to the design of a composite breakwater under breaking and non breaking wave conditions.

Key Words: Decomposition techniques, Benders decomposition, Cost optimization, Reliability based design, Failure probability, Modes of failure.

1 Introduction

Engineering design of structural elements is a complicated and highly iterative process that usually requires an extensive experience. Iterations consist of a trial-and-error selection of the design variables or parameters, together with a check of the safety and functionality constraints, until reasonable structures, in terms of cost and safety, are obtained. Since maintenance and repair take place during the service lifetime of the structure, the associated costs must be added to construction costs. The objective of the design is to verify that the structure satisfies the project requirements during its lifetime in terms of acceptable failure rates and cost (see Losada [1] and ROM [2]).

Since repair depends on the modes of failure and their frequencies, these must be defined. Each mode of failure m is defined by its corresponding limit state equation as, for example:

$$g_m(x_1, x_2, \dots, x_n) = h_{sm}(x_1, x_2, \dots, x_n) - h_{fm}(x_1, x_2, \dots, x_n); \quad m \in M, \tag{1}$$

where $(x_1, x_2, \dots, x_n)$ refer to the values of the variables involved, $g_m(x_1, x_2, \dots, x_n)$ is the safety margin and $h_{sm}(x_1, x_2, \dots, x_n)$ and $h_{fm}(x_1, x_2, \dots, x_n)$ are two opposing magnitudes (such as stabilizing and mobilizing forces, strengths and stresses, etc.) that tend to avoid and produce the associated mode of failure, respectively, and M is the set of all failure modes.

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The failure occurs when the critical variables satisfy $g_m \leq 0$. With the consideration of all extreme events (see Galambos [3] and Castillo [4]) that may occur in the reference period, the different failure rates for all failure modes can be estimated.

Over the last few years design methods have been improved by applying optimization techniques. The main advantage is that these techniques lead to optimal design and automation. Designer's concerns are only the constraints to be imposed on the problem and the objective function.

Some authors consider the construction cost (Castillo *et al.* [5, 6, 7, 8, 9]) or the total cost (construction, maintenance and repairs) as the design criteria (Van Dantzig [10], Burchart *et al.* (1995), Voortman *et al.* [11], Enevoldsen [12], and Enevoldsen and Sorensen [13, 14]). The main problem of including repair and maintenance cost is that in such a case the cost function includes yearly failure rates, the calculation of which implies solving as many optimization problems as failure modes. Thus, use of optimization programs is not straightforward.

In addition to requiring optimal solutions to problems, some interest is shown by people in knowing how sensitive are the solutions to data values. A sensitivity analysis provides excellent information on the extent to which a small change in the parameters or assumptions (data) modifies the resulting design (geometric dimensions, costs, reliabilities, etc.).

The aims of this paper are: (a) to present a decomposition design method that permits solving the total cost minimization problem, and (b) to provide tools to perform a sensitivity analysis.

The paper is structured as follows. In Section 2 the proposed method for optimal design based on Benders' decomposition is presented. In Section 3 a technique for performing a sensitivity analysis is explained. Section 4 illustrates the proposed method by an example dealing with the design of a composite breakwater. Section 5 is devoted to the discussion of the statistical assumptions. Section 6 presents a numerical example. Finally, Section 7 gives some conclusions.

2 Proposed Method for Optimal Design

In the design and reliability analysis of a structure, there are some random variables $(X_1, \ldots, X_n)$ involved. They include geometric variables, material properties, loads, etc. In this paper, without loss of generality, we make no distinction between random and deterministic variables. So, deterministic variables are only particular cases of them.

It is important to distinguish between design values of the random variables X_i, and actual values x_i $(i = 1, 2, \cdots, n)$. The design values are those values used by the engineer at the design stage for the geometric variables (dimensions), the material properties (strengths, stiffness, etc.), that do not necessarily correspond with those in the real work. Thus, in this paper the design values are assumed to be the means or the characteristic values (extreme percentiles) of the corresponding random variables, and are denoted $\bar{x}_i$ (mean) and $\tilde{x}_i$ (characteristic), respectively. Some of these design values are chosen by the engineer or given by the design codes, and some are selected by the optimization procedure to be presented. In this paper, the set of variables $(X_1, \ldots, X_n)$ will be partitioned in four sets:

1. **Optimized design variables $\boldsymbol{d}$:** Their mean values are to be chosen by the optimization procedure. Normally, they describe the dimensions of the work being designed, such as width, thickness, height, cross sections, etc., but can include material properties, etc.

2. **Non-optimized design variables $\boldsymbol{\eta}$:** Their mean or characteristic values are fixed by the engineer or the code guidelines as input data to the optimization program. Some examples are costs, material properties (unit weights, strength, Young modulus, etc.), and other geometric dimensions of the work being designed (bridge length, platform width, etc.) that are fixed.

3. **Random model parameters $\boldsymbol{\kappa}$:** Set of parameters used in the probabilistic design, defining the random spatial and temporal variability and dependence structure of the random variables involved (standard deviations, variation coefficients, correlations, covariance matrices, etc.).

4. **Dependent or non-basic variables $\boldsymbol{\psi}$:** Variables which can be written in terms of the basic variables $\boldsymbol{d}$ and $\boldsymbol{\eta}$ to facilitate the calculations and the statement of the problem constraints.

The corresponding means of $\boldsymbol{d}$ will be denoted $\bar{\boldsymbol{d}}$, and the mean or the characteristic values of $\boldsymbol{\eta}$ is denoted $\tilde{\boldsymbol{\eta}}$.

Given a set of values of the optimized design variables $\bar{\boldsymbol{d}}$, the probability of failure p_{st}^m under mode m during a random extreme event can be calculated using the joint probability density function $f(\boldsymbol{x}) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n; \boldsymbol{\theta})$ of all variables involved, by means of the integral:

$$p_{st}^m(\boldsymbol{\theta}) = \int\limits_{g_m(x_1,x_2,\ldots,x_n)\leq 0} f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n; \boldsymbol{\theta}) dx_1 dx_2 \ldots dx_n, \tag{2}$$

where $\boldsymbol{\theta}$ is a parametric vector which contains the parameters of the joint pdf of the variables $(X_1, X_2, \ldots, X_n)$. In this paper we assume that the parametric vector $\boldsymbol{\theta} = (\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$ contains the means $\bar{\boldsymbol{d}}$, the means or the characteristic values $\tilde{\boldsymbol{\eta}}$, and the vector of random model parameters $\boldsymbol{\kappa}$.

2.1 Model assumptions

Our model is based on the following assumptions:

1. The reference period for evaluating the probability of failure and, therefore, the repair costs will be taken as one year.

2. Failures occur only during extreme events, that are assumed to be stochastic processes, i.e., to occur at random times with yearly rate r_{st} (mean number of extreme events per year). Note that no assumption is needed about the dependence or independence of extreme events or the distribution of occurrence times, because only the yearly failure rate is sought.

3. The probability of failure in mode m in a random extreme event is $p_{st}^m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$. Thus, the mean number of failures per year (failure rate) is:

$$r_m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = r_{st} p_{st}^m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}); \quad m \in M. \tag{3}$$

4. One extreme event can cause at most only one failure of each type (mode) because repair is not possible during extreme events. This implies that failure accumulation is not included.

5. The proposed approach is based on guaranteeing bounded yearly failure rates of all failure modes. However, for the global failure rate, one can consider the well known lower and upper bounds $P_f^{lower} = \max\limits_m P_{f_m}$ and $P_f^{upper} = 1 - \prod\limits_{m=1}^{M} (1 - P_{f_m})$, respectively.

2.2 Total expected cost function

In this paper the criteria for design are based on minimizing the expectation of the total cost. The objective function consists of two components that describe the construction costs as a function of design variables, and the expected costs of failure (see Sorensen *et al.* [15], Voortman *et al.* [11]). Thus, expectation of the total cost, including construction and repairs, measured at initiation is:

$$E[cost] \;=\; C_0(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) + \sum_{n=1}^{D} \left(\sum_{m=1}^{M} C_{m0}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) r_m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \alpha_m^n \right), \tag{4}$$

where $E[cost]$ is the expected cost during the service lifetime D, $\bar{\boldsymbol{d}}$ and $\tilde{\boldsymbol{\eta}}$ are the design variables at their means and characteristic values, respectively, $C_0(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})$ is the initial construction cost, which could include design costs as well, $C_{m0}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})$ is the mean repairing cost associated with failure mode m, both evaluated at time $t = 0$, $r_m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$ is the mean number of failures per year related to mode m and α_m is the factor for interest and inflation correction of C_{m0} with respect to $t = 0$, that is equal to

$$\alpha_m = \left(\frac{1+f}{1+r} \right), \tag{5}$$

where f is the inflation rate and r is the interest rate.

2.3 Evaluation of the failure mode probabilities in an extreme event

In this paper we evaluate the failure mode probabilities in a random extreme event using first order reliability methods (FORM). More precisely, $p_{st}^m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$ for $m = 1, 2, \ldots, M$ is obtained using:

$$p_{st}^m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \underset{\boldsymbol{d}, \boldsymbol{\eta}}{\text{Maximum}} \quad \Phi(-\beta_m) = \Phi(-\sqrt{\boldsymbol{z}^T \boldsymbol{z}}) \;, \tag{6}$$

i.e., maximizing with respect to $\boldsymbol{d}, \boldsymbol{\eta}$, subject to

$$\boldsymbol{z} = \mathbf{G}(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}, \boldsymbol{\theta}) \tag{7}$$

$$\boldsymbol{q}(\boldsymbol{d}, \boldsymbol{\eta}) = \boldsymbol{\psi} \tag{8}$$

$$g_m(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}) = 0, \tag{9}$$

where β_m is the reliability index for failure mode m, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable, $G(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}, \boldsymbol{\theta})$ is the transformation leading to standard normal $\boldsymbol{z}$ variables used in FORM, $\boldsymbol{q}(\boldsymbol{d}, \boldsymbol{\eta}) = \boldsymbol{\psi}$ are the equations for the intermediate variables $\boldsymbol{\psi}$, and $g_m(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}) = 0$ is the boundary of the failure region for failure mode m.

Note that minimizing β_m is equivalent to a maximum likelihood estimation of a critical point in the failure surface in the standard normal space or to maximizing the probability of failure $\Phi(-\beta)$, because the $\Phi()$ function (cdf of the standard normal distribution) is increasing. In this paper the last alternative has been used because the sensitivities of the probabilities of failure with respect to the data are sought after, and this approach leads to direct formulas.

For a complete description of "First Order Reliability Methods" (FORM) and some examples see Madsen, Krenk and Lind [16], Ditlevsen and Madsen [17], or Melchers [18].

Once the probabilities for all failure modes have been calculated it is possible to obtain the repair costs C_{m0} as a function of the failure rate (see (3)).

Then, we are ready to state the design problem as an optimization problem as follows.

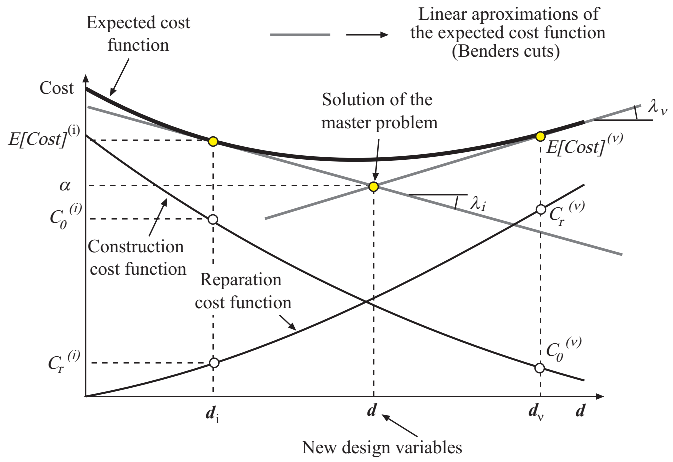

Figure 1: Graphical illustration of how the expected cost function is approximated using Benders cuts.

2.4 Design as an optimization problem

In this paper the design problem is equivalent to solving the following optimization problem:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \quad E[cost] = C_0(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) + \sum_{n=1}^{D} \left(\sum_{m=1}^{M} C_{m0}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) r_m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \alpha_m^n \right) , \tag{10}$$

subject to

$$\boldsymbol{q}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) = \boldsymbol{\psi} \tag{11}$$

$$\mathbf{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) \leq \mathbf{0}, \tag{12}$$

where $\mathbf{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) \leq \mathbf{0}$ is the set of geometric or design constraints, $r_m(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$ is given by solving the problem (6)-(9), which is a FORM solution for p_{st}^m, and using (3).

2.5 Solving the cost optimization problem using Benders' decomposition

This type of problem can be solved using decomposition techniques (see Benders [19] and Geoffrion [20]) that were applied to reliability optimization problems by Mínguez *et al.* [21]. The price that has to be paid for such a simplification is iteration. That is, instead of solving the original problem at once, two simpler problems are solved iteratively: a simple problem called master problem (approximation of the original one) and the subproblem or subproblems (evaluation of the probability of failure using FORM), so that subproblems are being progressively taken into account into the master problem. The expected cost function is approximated by an increasing number of hyperplanes (see Figure 1). Note that for fixed values of the design variables $\bar{\boldsymbol{d}}$ the function to be optimized can be evaluated solving the optimization problems (6)-(9) and using (3), so in the following these variables are considered as complicating (design) variables.

The following iterative scheme can be applied to solve the problem (10)-(12).

- **Step 0: Initialization.** Initialize the iteration counter $\nu = 1$, α to its initial lower bound α^{lo} and select some initial values for the design variables $\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}}_1$.

- **Step 1: Subproblem solution.** Solve the subproblems, i.e., the problems (6)-(9) modified to

$$r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \underset{\boldsymbol{d}, \boldsymbol{\eta}, \bar{\boldsymbol{d}}}{\text{Maximum}} \quad r_{st}\Phi(-\beta_m) = r_{st}\Phi(-\sqrt{\boldsymbol{z}^T\boldsymbol{z}}) \tag{13}$$

subject to

$$\begin{aligned} \boldsymbol{z} &= \mathbf{G}(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}, \bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) && (14)\\ g_m(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}) &= 0, && (15)\\ \boldsymbol{q}(\boldsymbol{d}, \boldsymbol{\eta}) &= \boldsymbol{\psi} && (16)\\ \bar{\boldsymbol{d}} &= \bar{\boldsymbol{d}}_\nu \ : \ \boldsymbol{\mu}_{m\nu}, && (17) \end{aligned}$$

where the notation in (17) is used to refer to the dual variables $\boldsymbol{\mu}_{m\nu} = \partial r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})/\partial \bar{\boldsymbol{d}}_\nu$ associated with the constraint $\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}}_\nu$. Note that dual variables are the sensitivities of the objective function (13) with respect to $\bar{\boldsymbol{d}}_\nu$.

Next, evaluate the cost function $\alpha(\bar{\boldsymbol{d}}_\nu)$:

$$\alpha(\bar{\boldsymbol{d}}_\nu) = C_0(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) + \sum_{n=1}^{D} \left(\sum_{m=1}^{M} C_{m0}(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \alpha_m^n \right), \tag{18}$$

where $\alpha(\bar{\boldsymbol{d}}_\nu)$ is the total cost for fixed values of the complicating (optimizing design) variables (see the corresponding point of the total cost function in Figure 1).

Finally, for approximating the objective function (total expected cost) in (10), we need to obtain the derivatives of the objective cost function using the formula:

$$\boldsymbol{\lambda}_\nu = \frac{\partial C_0(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})}{\partial \bar{\boldsymbol{d}}_\nu} + \sum_{n=1}^{D} \left(\sum_{m=1}^{M} \left[\frac{\partial C_{m0}(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})}{\partial \bar{\boldsymbol{d}}_\nu} r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) + C_{m0}(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) \boldsymbol{\mu}_{m\nu} \right] \alpha_m^n \right), \tag{19}$$

where $\boldsymbol{\mu}_{m\nu} = \partial r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})/\partial \bar{\boldsymbol{d}}_\nu$ are the derivatives of the yearly failure rates $r_m(\bar{\boldsymbol{d}}_\nu, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})$ obtained as the dual variables of the problem (13)-(17).

- **Step 2: Convergence checking.** As function (18) is more constrained than the objective function of the original problem (10)-(3) in the sense that the optimization variables are fixed for the actual iteration, an upper bound of the objective function optimal value is computed as $z_{\text{up}}^{(\nu)} = \alpha(\bar{\boldsymbol{d}}_\nu)$. A lower bound of the objective function optimal value is $z_{\text{down}}^{(\nu)} = \alpha$. If $\left| \frac{z_{\text{up}}^{(\nu)} - z_{\text{down}}^{(\nu)}}{z_{\text{up}}^{(\nu)}} \right|$ is lower than the tolerance, the procedure stops, otherwise, it goes to **Step 3**.

- **Step 3: Master problem solution for iteration ν.** The master problem is solved:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \quad \alpha \tag{20}$$

subject to

$$\begin{aligned} \alpha &\geq \alpha(\bar{\boldsymbol{d}}_k) + \boldsymbol{\lambda}_k^T(\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}_k); \quad k = 1, 2, \cdots, \nu - 1 && (21)\\ \boldsymbol{q}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) &= \boldsymbol{\psi} && (22)\\ \mathbf{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) &\leq \mathbf{0} && (23) \end{aligned}$$

$$\alpha \geq \alpha^{\mathrm{lo}}, \tag{24}$$

obtaining $\bar{\boldsymbol{d}}_\nu$ (see the intersection of the hyperplanes in Figure 1). Note that the objective function of the original problem in (10) is replaced by the linear approximations in (21) (see the Benders cuts in Figure 1). As problem (20)-(24) is a relaxation of the original problem (10)-(3) the optimal solution of this problem (α) is a lower bound of the objective function optimal value.

Let $\nu = \nu + 1$, and go to Step 1 and the process is repeated until convergence.

3 Sensitivity Analysis

The problem of sensitivity analysis in reliability based optimization has been discussed by several authors, see, for example, Enevoldsen [22]. In this section we show how the duality methods can be applied to sensitivity analysis in a straightforward manner. We emphasize here that the method to be presented in this section is of general validity.

When the parameters with respect to which the sensitivities are looked for appear on the right hand side of one of the constraints in an optimization problem, then, the corresponding sensitivities are simply the values of the associated dual variables, that practically all software optimization packages give by free because once the optimal solution has been found, they can be easily calculated. The problem arises when the data or parameters with respect to which we want to calculate the sensitivities do not appear on the right hand side of a constraint.

One way of solving this problem consists of generating auxiliary (redundant) constraints that satisfy such a condition. One way of generating these constraints consists of transforming all the parameters or data with respect to which the sensitivities are sought for, into auxiliary variables and adding the constraints that set the variables to their actual values. To illustrate, we apply this technique to the optimization problems (13)-(17) and (20)-(24) at the optimal solution $\bar{\boldsymbol{d}}^*$.

The problem (13)-(17) is obviously equivalent to the problem

$$r_m(\bar{\boldsymbol{d}}^*, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \underset{\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{d}^*, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*}{\text{Maximum}} \quad r_{st}\Phi(-\beta_m) = r_{st}\Phi(-\sqrt{\boldsymbol{z}^T\boldsymbol{z}}) \tag{25}$$

subject to

$$\begin{aligned}
\boldsymbol{z} &= \mathbf{G}(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*, \boldsymbol{d}^*) && (26)\\
\boldsymbol{q}(\boldsymbol{d}, \boldsymbol{\eta}) &= \boldsymbol{\psi} && (27)\\
g_m(\boldsymbol{d}, \boldsymbol{\eta}, \boldsymbol{\psi}) &= 0 && (28)\\
\boldsymbol{d}^* &= \bar{\boldsymbol{d}}^* : \boldsymbol{\mu}_m && (29)\\
\boldsymbol{\eta}^* &= \tilde{\boldsymbol{\eta}} : \boldsymbol{\delta}_m && (30)\\
\boldsymbol{\kappa}^* &= \boldsymbol{\kappa} : \boldsymbol{\xi}_m, && (31)
\end{aligned}$$

where $\boldsymbol{d}^*$, $\boldsymbol{\eta}^*$ and $\boldsymbol{\kappa}^*$ are the auxiliary variables.

The basic idea is simple. Assume that we wish to know the sensitivity of the objective function to changes in some data values $\bar{\boldsymbol{d}}^*$, $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$. Converting the data into auxiliary variables, $\boldsymbol{d}^*$, $\boldsymbol{\eta}^*$ and $\boldsymbol{\kappa}^*$, and setting them, by means of constraints (29)-(31), to their actual values $\bar{\boldsymbol{d}}^*$, $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$, we obtain a problem that is equivalent to the initial optimization problem but has constraints such that the values of the dual variables associated with them give the desired sensitivities. More precisely, the values of the dual variables $\boldsymbol{\mu}_m$, $\boldsymbol{\delta}_m$ and $\boldsymbol{\xi}_m$ associated with constraints (29)-(31) give

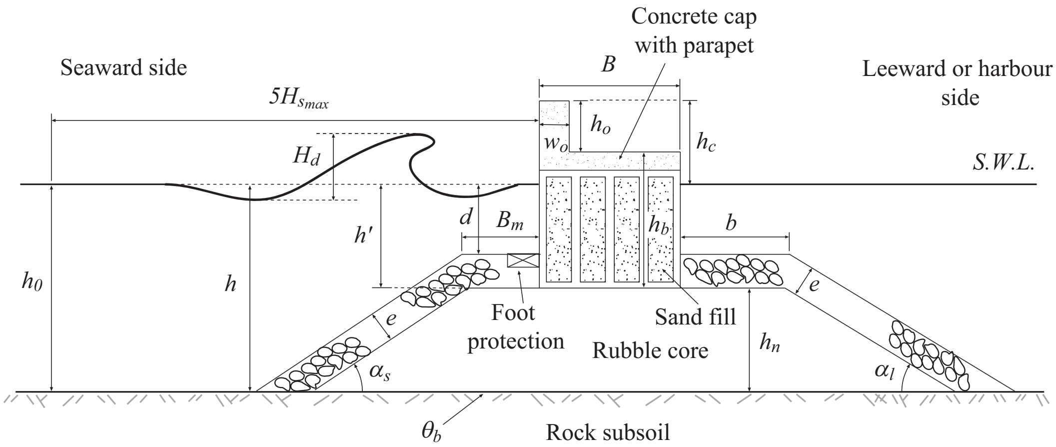

Figure 2: Composite breakwater showing the geometric design variables.

the sensitivities of the yearly failure rates to $\bar{\boldsymbol{d}}^*$, $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$, respectively. These sensitivities allow us determining for example how the reliability of an engineering design changes when its design values and the statistical parameters of the random variables involved are modified.

Similarly, the problem (20)-(24) is obviously equivalent to the problem

$$\underset{\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*}{\text{Minimize}} \quad \alpha \tag{32}$$

subject to

$$\alpha \geq \alpha(\bar{\boldsymbol{d}}_j) + \boldsymbol{\lambda}_j^T(\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}_j) + \boldsymbol{\phi}^T(\boldsymbol{\eta}^* - \tilde{\boldsymbol{\eta}}) + \boldsymbol{\epsilon}^T(\boldsymbol{\kappa}^* - \boldsymbol{\kappa}); \;\; j \in J \tag{33}$$

$$\boldsymbol{q}(\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*) = \boldsymbol{\psi} \tag{34}$$

$$\mathbf{h}(\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*, \boldsymbol{\psi}) \leq \mathbf{0} \tag{35}$$

$$\alpha \geq \alpha^{\text{lo}} \tag{36}$$

$$\boldsymbol{\eta}^* = \tilde{\boldsymbol{\eta}} \tag{37}$$

$$\boldsymbol{\kappa}^* = \boldsymbol{\kappa}, \tag{38}$$

where now $\boldsymbol{\eta}^*$ and $\boldsymbol{\kappa}^*$ are the auxiliary variables and J is the set of active Benders cuts at the optimal solution.

Note that the variables $\boldsymbol{\phi}$ and $\boldsymbol{\epsilon}$ are the partial derivatives of the total expected cost function with respect $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$, respectively. The values of the dual variables associated with constraints (37) and (38) give the sensitivities of the total expected cost to $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$, respectively. These sensitivities allow us determining how the expected cost of a breakwater changes when the geometric dimensions and the statistical parameters of the random variables are modified.

4 Optimized Design of a Composite Breakwater

The probability based design of composite breakwaters has been studied by Christiani *et al.* [23], Burcharth and Sorensen [24], Sorensen and Burcharth [25], as well as in the European project PROVERBS (see Oumeraci *et al.* [26]) and the PIANC Working Group 28 on Breakwaters with Vertical and Inclined Concrete Walls [27].

In this section the proposed procedure is applied to the design of a composite breakwater. The main section of the breakwater is shown in Figure 2 where the main parameters are shown. Notice that these parameters define geometrically the different elements of the cross section and must be defined in the construction drawings. Our goal is an optimal design based on minimizing the construction and repair costs.

4.1 Model assumptions

The breakwater example to be discussed below is simply an illustrative example, and it must be considered as such because the analysis cannot be considered exhaustive, since several failure modes were not implemented. Our breakwater model is based on the following assumptions:

1. The extreme events will be storms.

2. Long-term statistics for storms are characterized by a set of three variables ($H_{s_{max}}$,H_{max}, and $T_{z_{max}}$). The maximum significant wave height $H_{s_{max}}$ of all its sea states (H_s is a parameter of a sea state defined as the mean of the one third highest wave heights in the sea state) is used to characterize the severity of the storm. H_{max} is its maximum wave height (the peak) within the sea state associated with $H_{s_{max}}$, and $T_{z_{max}}$ is the wave period occurring with H_{max}. It is assumed that they are dependent random variables whose probability distribution and dependence structure must be derived from real data. Once a storm has occurred, its intensity and characteristics can be derived from this joint distribution, i.e., a set of values $\{H_{s_{max}}, H_{max}, T_{z_{max}}\}$ can be drawn at random from a population with the corresponding distribution.

3. Interaction between failure modes though an important problem is not considered here.

4.2 Modes of failure

In this study, a total of 8 modes of failure have been considered: sliding (s), turning (t), 4 foundation (b, c, d, sea), overtopping (o), and seaside berm instability (a) failures (see Figure 3).

The external wave forces on the upright section are the most important considerations in the design of vertical breakwaters, including both pulsating and impact wave loads. The well known Goda pressure formulas (see Goda [28]) for the evaluation of the forces acting on the breakwater (see Figure 3) have been used in this paper. But as the impulsive pressure coefficient used in Goda's formula does not accurately estimate the effective pressure due to impulsive pressure under all conditions the new impulsive pressure coefficient proposed by Takahashi *et al.* [29] is used. The maximum wave height (H_{break}) is adjusted in the surf zone due to random wave breaking as described by Goda [30]:

$$\frac{H_{break}}{L_0} = A\left\{1 - \exp\left(-1.5\frac{\pi h_0}{L_0}(1 + 15\tan^{4/3}\theta_b)\right)\right\}, \tag{39}$$

where h_0 is the water height in the distance of five times the maximum significant wave height $H_{s_{max}}$ toward the offshore of the breakwater, L_0 is the deep water wave length, θ_b is the mean angle of the sea bottom, and the coefficient A takes different values depending of the kind of waves, it takes the value 0.17 for regular waves. Its upper and lower limits are 0.18 and 0.12, respectively.

Thus the design wave height H_d is

$$H_d = \min(H_{max}, H_{break}). \tag{40}$$

Sliding failure. This failure occurs when the breakwater caisson suffers an horizontal displacement, it can occur as a slip either at the interface between the caisson concrete base and the rubble material, or entirely in the rubble material. The safety against sliding failure can be verified by the following limit state equation (see Figure 3(a)):

$$g_s = \min(\mu_c, \tan(\phi_r))(W_1 - F_v) - F_h, \tag{41}$$

where μ_c is the friction coefficient, ϕ_r is the angle of internal friction of rubble, F_h and F_v are the total vertical and horizontal forces due to wave pressure, and W_1 is the actual caisson weight reduced for buoyancy, which are given by:

$$\begin{aligned} F_h &= h_c(p_1 + p_4)/2 + h'(p_1 + p_3)/2 && (42)\\ F_v &= \frac{1}{2} p_u B && (43)\\ W_1 &= V_c\gamma_c - h'B\gamma_w && (44)\\ V_c &= Bh_b + w_o h_o, && (45) \end{aligned}$$

where h_c is the freeboard, p_1, p_3 and p_4 are the Goda's pressures at the water level, caisson's bottom and freeboard, respectively, p_u is the uplift pressure, B the caisson width, V_c is the total caisson volume, γ_c is the average unit weight of caisson, h' is submerged height of the caisson, γ_w is the water unit weight, h_b is caisson height, and h_o and w_o are the parapet breakwater height and width, respectively.

Overturning failure. This failure occurs when the breakwater structure rotates with respect to point O (see Figure 3(b)) because of water pressure forces. The safety against turning failure can be verified by the following limit state equation:

$$g_t = W_1 y - M_v - M_h, \tag{46}$$

where y is the W_1 offset with respect to point O, and M_v and M_h are the moments with respect to point O of the vertical and horizontal water pressure forces, which are given by

$$M_v = \frac{2}{3} F_v B = \frac{1}{3} p_u B^2 \tag{47}$$

and

$$M_h = \frac{1}{6}(2p_1 + p_3)h'^2 + \frac{1}{2}(p_1 + p_4)h'h_c + \frac{1}{6}(p_1 + 2p_4)(h_c)^2. \tag{48}$$

Foundation failure. The following geotechnical failure functions for a feasibility level of sophistication proposed by Oumeraci *et al.* [26] considering that the subsoil material is rock are used in this paper:

1. Rotation failure (b).
2. Rupture surface through rubble only (c).
3. Rupture surface through rubble and along top of subsoil (d).
4. Additionally, we have also considered the seaward rupture surface through rubble only (sea).

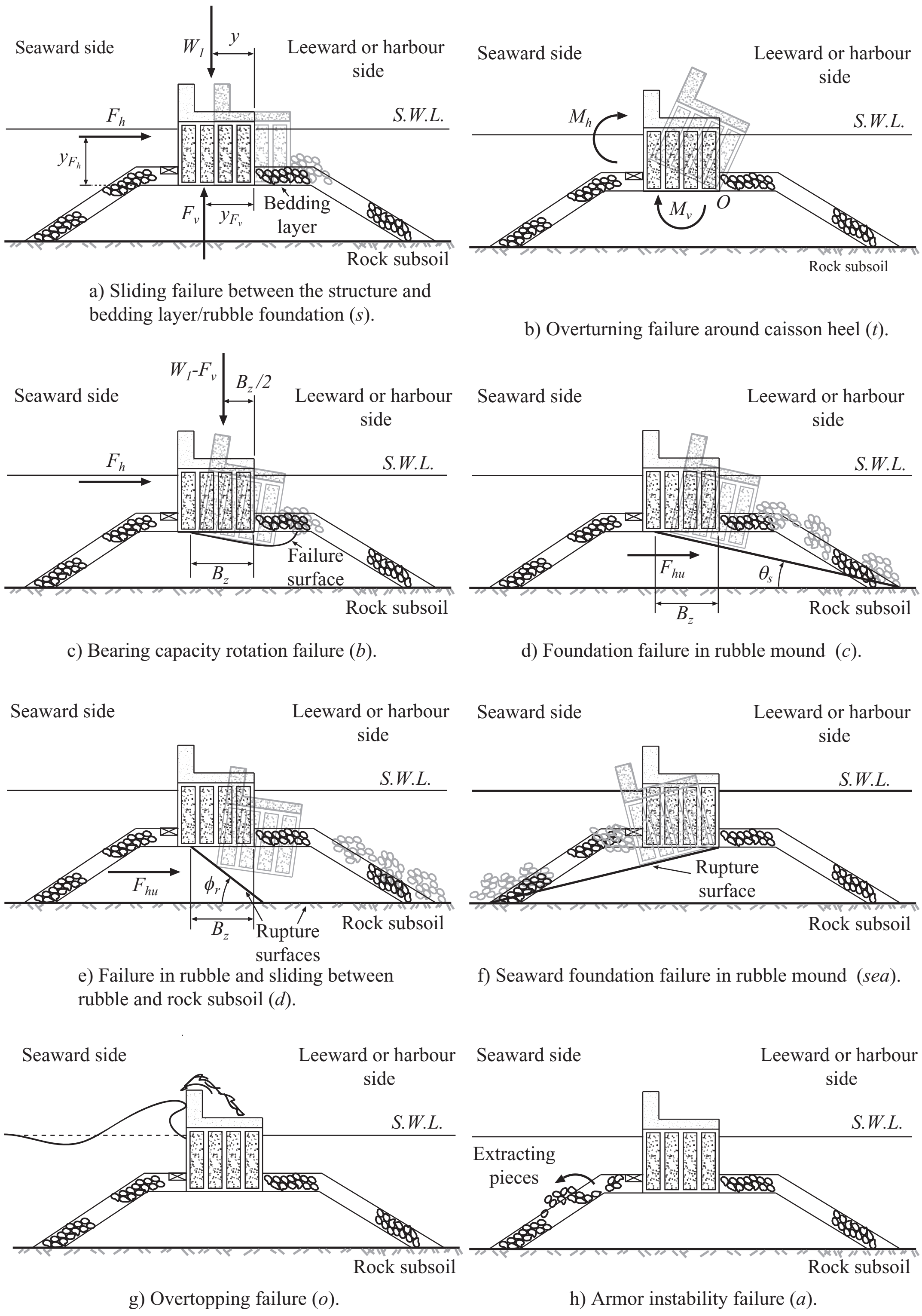

Figure 3: The eight composite failure modes considered in the breakwater example.

The set of failure modes consists of a limited number of failure surfaces with a known a-priori geometry (see Figures 3 (c)-(f)). Alternatively, more sophisticated equations based on the upper bound theory can be used (see Sorensen and Burchart [25], Oumeraci *et al.* [26]).

It is often very practical to consider the equilibrium of the wall separately from the equilibrium of the soil, thus the integrated effective stresses acting on the skeleton of rubble foundation are obtained as resultant from the other forces acting on the wall. The distance of the vertical force $W_1 - F_v$ component to the harbour side edge B_z is:

$$B_z = 2\frac{W_1 y - F_h y_{F_h} - F_v y_{F_v}}{W_1 - F_v}, \tag{49}$$

where y_{F_h} and y_{F_v} are the lever arms of F_h and F_v, respectively.

The effect of wave induced pressure along the rupture boundary inside the rubble (F_{hu}) can be obtained under the assumptions of triangular pressure distribution in the horizontal direction and hydrostatic pressure in the vertical direction as:

$$F_{hu} = \begin{cases} \dfrac{B_z^2 \tan\theta_s}{2B} p_u & \text{if} \quad B_z \leq h_n/\tan\theta_s, \\ \dfrac{h_n(2B_z - h_n/\tan\theta_s)}{2B} p_u & \text{if} \quad B_z > h_n/\tan\theta_s, \end{cases} \tag{50}$$

where h_n is the core height, and θ_s is the angle between the bottom of the wall and the rupture surface (see Figure 3 (d)), that can be obtained as:

$$\theta_s = \arctan\frac{h_n}{B_z + b + (h_n + e)\cot\alpha_\ell}, \tag{51}$$

where b is the leeward berm width, e is the armor layer thickness and α_ℓ is leeward slope angle.

Then, the safety against rotation failure can be verified by the following limit state equation (see Figure 3 (c)):

$$g_b = B_z^2(\gamma_s - \gamma_w)\tan\phi_r\left(\tan^2(\pi/4 + \phi_r/2)\exp(\pi\tan\phi_r) - 1\right) - (W_1 - F_v)\left(\frac{1}{1 - F_h/(W_1 - F_v)}\right)^3, \tag{52}$$

where γ_s is the rubble mound unit weight.

The safety against rupture surface through rubble only failure can be verified by the following limit state equation (see Figure 3 (d)):

$$g_c = W_1 - F_v + (\gamma_s - \gamma_w)[(B_z + b + e\cot\alpha_\ell)h_n/2 + (b + e\cot\alpha_\ell/2)e] - (F_h + F_{hu})\cot(\phi_r - \theta_s). \tag{53}$$

The safety against rupture surface through rubble and along top of subsoil failure can be verified by the following limit state equation (see Figure 3 (e)):

$$\begin{aligned} g_d \;=\; & (W_1 - F_v + (\gamma_s - \gamma_w)\left[(2(B_z + b + e\cot\alpha_\ell) + h_n(\cot\alpha_\ell - \cot\alpha_{\phi_r}))h_n/2\mu_s \right. \\ & \left. + (b + e\cot\alpha_\ell/2)e]\right) - (F_h + F_{hu}), \end{aligned} \tag{54}$$

where μ_s is the friction coefficient between the rubble bedding layer and the rock subsoil. Note that in this case the angle between the bottom of the wall and the rupture surface is ϕ_r (see Figure 3 (e)), In addition, for avoiding the seaward failure in calm sea wave conditions, the rupture surface through rubble only is considered using the following limit state equation (see Figure 3 (f)):

$$g_{sea} = \phi_r - \arctan\left(\frac{h_n}{B + B_m + (e + h_n)\cot\alpha_s)}\right), \qquad (55)$$

where B_m is the seaward berm width and α_s is the seaward slope angle. Note that no wave forces are considered in this failure mode, so the yearly probability treatment in the lifetime of the structure will be different than the other failure modes (it does not depend on r_{st}).

Overtopping failure. For a composite breakwater of seaboard slope $\tan\alpha_s$ and freeboard h_c, (see Figure 3 (g)), and a sea state defined by a significant maximum wave height $H_{s_{max}}$, the mean overtopping volume q per unit of breakwater length is given, for a caisson breakwater, by the exponential relation (see Franco and Franco [31])

$$q = a\exp(-b_o h_c/H_{s_{max}})\sqrt{gH^3_{s_{max}}}, \qquad (56)$$

where $q/\sqrt{gH^3_{s_{max}}}$ is the dimensionless discharge, $h_c/H_{s_{max}}$ is the relative freeboard, and a and b_o are coefficients that depend on the structure shape and on the water surface behavior at the seaward face.

The safety against overtopping failure can be verified from the following equation:

$$g_o = q_0 - q, \qquad (57)$$

where q_0 is the maximum allowable mean overtopping discharge for structural damage.

Berm instability failure. It is customary in caisson breakwater construction to provide a few rows of foot-protection concrete blocks at the front and rear of the upright section. It usually consists of rectangular blocks weighting form 100 to 400 kN depending on the design wave height. This protection is indispensable especially against oblique wave attack. The remainder of the berm and slope of the rubble mound foundation must be protected with armor units of sufficient weight to withstand the wave action. In this paper berm instability failure refers to the removal of pieces from the berm and slope as it is shown in Figure 3 (h).

Based on experiments, Losada [1] and following Tanimoto, Yagyu and Goda [32], proposed the following limit state equation to evaluate the dimensionless quantity $\dfrac{W}{\gamma_w H_d^3}$:

$$\frac{W}{\gamma_w H_d^3} = R\Phi_e, \qquad (58)$$

where Φ_e is the berm stability function, R is an dimensionless constant, which depends on γ_s (for rubble armor units) and γ_w, and W is the individual armor block weight of the berm, that are given by

$$W = \gamma_s \ell_e^3 \qquad (59)$$

$$R = \frac{\gamma_s/\gamma_w}{\left(\dfrac{\gamma_s}{\gamma_w} - 1\right)^3} \qquad (60)$$

$$\Phi_e = \min\left\{0.3, \left[4.2\frac{(1-c)d}{c^{1/3}H_d} + 3.24\exp\left(-2.7\frac{d(1-c)^2}{H_d c^{1/3}}\right)\right]^{-3}\right\} \quad (61)$$

$$c = \frac{4\pi d}{L\sinh\left(\dfrac{4\pi d}{L}\right)}\sin^2\left(\frac{2\pi B_m}{L}\right), \quad (62)$$

where l_e is the equivalent cubic block side, d is the berm depth in front of the caisson, c is an auxiliary variable, and B_m is the seaward width. Under such a conditions, the occurrence of failure can be determined from the following equation:

$$g_a = W - \gamma_w R\Phi_e H_d^3. \quad (63)$$

4.3 Practical design criteria

In maritime works there are some rules of good practice that should be observed. Some of them are country dependent and some have historical roots, others are taken as a precaution against impulsive breaking wave conditions. Those used in this example, are:

1. **Layers slopes and berms widths**: The seaside and leeward berm and slope protection has the following restrictions. The minimum armor unit weight allowed is 0.3 kN while the maximum is 21 kN (concrete pieces have to be used for greater weight armor units), this implies that the armor layer thickness limits are ($e = 2\ell_e$):

$$0.5 \le e \le 2 \ (m), \quad (64)$$

   where ℓ_e is the equivalent cubic block side for the main layer. The minimum berm widths limits, note that berm widths in Spain are smaller than usual berm widths in Japan, are:

$$B_m \ge 2\ell_e; \qquad b \ge 2\ell_e. \quad (65)$$

   The maximum berm widths limits are:

$$B_m \le 1.5B; \qquad b \le 1.5B. \quad (66)$$

   The gradient of the slope of the rubble mound is usually set to

$$1.5 \le \cot\alpha_s \le 3; \qquad 1.5 \le \cot\alpha_\ell \le 3. \quad (67)$$

2. **Construction or operational reasons**: The caisson width limits are:

$$10 \le B \le 35 \ (m), \quad (68)$$

   while the maximum seaward (h_c) and the minimum leeward freeboard are

$$h_c = h_n + h_b + h_o - h_{lo} - t_r \le 15; \qquad h_n + h_b - h_{lo} - t_r \le 1 \ (m), \quad (69)$$

   respectively, where h_b is caisson height, h_n is the core height, h_{lo} is the water depth in front of the caisson corresponding to the zero port reference level (minimum water depth) and t_r is the tidal range.

The minimum water level in front of the vertical breakwater and the minimum water depth in front of the caisson are respectively,

$$h \geq h_{lo}; \qquad d \geq h_{lo} - (h_h + e). \tag{70}$$

For the vertical breakwater to be a composite breakwater we must have:

$$\frac{h_n + e}{h_{lo} + t_r} \geq 0.3; \qquad \frac{h_n + e}{h_{lo}} \leq 0.9. \tag{71}$$

3. **Geometric identities**:

$$h = h_1 + h_2; \qquad h = h' + h_n; \qquad h' + h_c = h_b + h_o; \qquad d + e = h', \tag{72}$$

where h_1 is the water level owing to the astronomical tide, and h_2 is the water level produced by barometrical or storm surge effects.

4.4 Construction cost

The details of the derivation of the cost function are given in Appendix A. The resulting total construction cost becomes:

$$C_0(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) = C_c V_c + C_{a\ell} V_{a\ell} + C_{co} V_{co}, \tag{73}$$

where $V_c, V_{a\ell}$ and V_{co} are the sand filled caissons, armor layer, and core volumes, respectively, and $C_c, C_{a\ell}$ and C_{co} are the respective construction costs per unit volume.

4.5 Repair cost

The repair cost C_{m0} for each mode of failure m has been assumed to be a fraction r_{p_m} of the construction costs, which depends on the consequences of the failure, thus

$$C_{m0} = r_{p_m} C_0. \tag{74}$$

4.6 Set of variables

As an illustration, for the composite breakwater example (see Figure 2), the optimized design variables ($\boldsymbol{d}$), the non-optimized design variables ($\boldsymbol{\eta}$), the random model parameters ($\boldsymbol{\kappa}$), and the dependent variables ($\boldsymbol{\psi}$) are, respectively:

$$\begin{aligned}
\boldsymbol{d} &= \{b, B, B_m, e, h_b, h_n, h_o, \alpha_\ell, \alpha_s\} \\
\boldsymbol{\eta} &= \{H_{max}, H_{s_{max}}, T_{z_{max}}, h_1, h_2, \theta_w, A, \mu_c, \mu_s, \phi_r, \gamma_c, \gamma_s, A_g, B_g, M_{A_g}, M_{B_g}, S_g, b_o, C_{ar}\} \\
&\quad \cup \{F_m, a, C_{a\ell}, C_c, C_{co}, D, f, q_0, r, r_{p_m}, r_{st}, w_o, \gamma_w, \tan\theta_b\} \\
\boldsymbol{\kappa} &= \{v_b, v_{B_m}, v_e, v_{h_n}, v_{\alpha_\ell}, v_{\alpha_s}, \kappa_s, \delta_s, \lambda_s, a_r, b_r, \kappa_w, \delta_w, \lambda_w, a_t, b_t, c_t, \sigma_{T_{z_{max}}}, h_{lo}, t_r, \sigma_{h_2}, \sigma_{\theta_w}\} \\
&\quad \cup \{\sigma_A, \sigma_{\mu_c}, v_{\mu_s}, v_{\phi_r}, \sigma_{\gamma_c}, \sigma_{\gamma_s}, \sigma_{A_g}, \sigma_{B_g}, \sigma_{M_{A_g}}, \sigma_{M_{B_g}}, \rho_{A_g}, \rho_{B_g}, \sigma_{S_g}, \sigma_{b_o}, \sigma_{C_{ar}}, v_{F_m}\} \\
\boldsymbol{\psi} &= \{B_z, c, d, F_h, F_v, F_{hu}, h, h_0, h', h_c, H_{break}, H_d, \ell_e, L, L_0, M_h, M_v, p_1, p_3, p_4, p_u\} \\
&\quad \cup \{q, R, V_c, V_{a\ell}, V_{co}, W, W_1, y, y_{F_h}, y_{F_v}, \alpha_m, \theta_s, \Phi_e\}.
\end{aligned}$$

5 Statistical assumptions

To complete the model, the statistical assumptions need to be provided. They are strongly dependent on the location of the maritime structure. For illustrative purposes, in this section we present those for a composite breakwater in the harbor at Gijón.

The service life of the breakwater is consider to be $D = 50$ years, and the joint distribution of all variables involved is based on the following assumptions:

1. Optimized design variables: The subset $\{B, h_b, h_o\}$ of optimized design variables $\boldsymbol{d}$ related to the concrete caisson are assumed to be deterministic because the construction control is good, whereas the subset of variables associated with the rubble mound $\{b, B_m, e, h_n, \alpha_\ell, \alpha_s\}$ are considered normal random variables whose mean values are obtained from the optimization procedure. In what follows the mean value, standard deviation and the coefficient of variation of any variable x will be denoted as μ_x, σ_x and v_x, respectively.

2. Load variables: The joint distribution of $(H_{s_{max}}, H_{max}, T_{z_{max}})$ which define our simplified storms and other factors affecting the incident waves, are defined by:

   (a) The marginal cumulative distribution function of $H_{s_{max}}$. Based on extreme value considerations and the truncated character of the simplified storms (they were considered for $H_{s_{max}} \geq 3$), $H_{s_{max}}$ can be assumed to be a generalized Pareto distribution:

   $$F_{H_{s_{max}}}(H_{s_{max}}) = 1 - \left(1 - \frac{\kappa_s(H_{s_{max}} - \lambda_s)}{\delta_s}\right)^{1/\kappa_s} \; ; \; 1 - \frac{\kappa_s(H_{s_{max}} - \lambda_s)}{\delta_s} \geq 0. \qquad (75)$$

   where κ_s, λ_s and δ_s are the parameters to be estimated from the data.

   (b) The conditional distribution $H_{max}|H_{s_{max}}$ of H_{max} given $H_{s_{max}}$. Based on a regression analysis it is observed that they exhibit a linear regression:

   $$\hat{H}_{max} = a_r + b_r \; H_{s_{max}}, \qquad (76)$$

   where $\hat{H}_{max}$ is the estimate H_{max} given $H_{s_{max}}$ and a_r and b_r are the linear regression coefficients, and that the residuals $H_{max} - \hat{H}_{max}$ follow a maximal Weibull model with cumulative distribution function:

   $$F_X(x) = \exp\left\{-\left[1 - \kappa_w\left(\frac{x - \lambda_w}{\delta_w}\right)\right]^{1/\kappa_w}\right\}; \quad 1 - \kappa_w\left(\frac{x - \lambda_w}{\delta_w}\right) \geq 0. \qquad (77)$$

   The combination of both assumptions leads to the final model for $H_{max}|H_{s_{max}}$.

   (c) The conditional distribution $T_{z_{max}}|H_{max}, H_{s_{max}}$ of $T_{z_{max}}$ given $H_{max}, H_{s_{max}}$. Based on a regression analysis the following model is obtained:

   $$\bar{T}_{z_{max}} = a_t + b_t \; H_{s_{max}} + c_t \; H_{max}, \qquad (78)$$

   where a_t, b_t and c_t are the linear regression coefficients, with residuals $T_{z_{max}} - \bar{T}_{z_{max}}$ following a normal distribution:

   $$T_{z_{max}} - \bar{T}_{z_{max}} \sim N(0, \sigma^2_{T_{z_{max}}}),$$

   where $\sigma_{T_{z_{max}}}$ is the standard deviation.
   The combination of these assumptions leads to the final model for $T_{z_{max}}|H_{max}, H_{s_{max}}$.

(d) The water depth h_1, considering the tidal elevation, is modelled as a random variable with cumulative distribution function:

$$F_{h_1}(x) = \frac{\arccos\left(2(h_{lo} - x)/t_r + 1\right)}{\pi}, \tag{79}$$

where h_{lo} is the minimum value of h_1 (zero port reference level) and t_r is the tidal range.

(e) The water rise level owing to meteorological causes h_2 is assumed to be a normal random variable with mean μ_{h_2} and standard deviation σ_{h_2}.

(f) The incident wave angle θ_w is assumed to be normal $N(0, \sigma^2_{\theta_w})$.

(g) The coefficient A in (39) for modelling the change in the maximum wave height due to random wave breaking is modelled as a normal random variable. As there is no clear information on the variance but only reasonable extreme values, the simple rule that two standards deviations account for the difference between the maximum (minimum) and the mean value was adopted. Thus, $\mu_A = (0.18 + 0.12)/2 = 0.15$ and $\sigma_A = (0.18 - 0.12)/4 = 0.015$.

3. The soil strength is modelled using the following assumptions:

(a) The friction factor μ_c between the caisson base and the rubble is assumed as log-normal distributed with mean μ_{μ_c} and standard deviation σ_{μ_c}.

(b) The friction coefficient μ_s between the rubble bedding layer and the rock subsoil is assumed as log-normal distributed with mean μ_{μ_s} and coefficient of variation v_{μ_s}.

(c) Since the breakwater foundation is made of friction material an statistical model for the angle of internal friction of rubble is required. This angle is modelled by a normal random variable with mean μ_{ϕ_r} and coefficient of variation v_{ϕ_r}. We do not take into account spatial variation.

(d) The average unit weights of caisson γ_c and rubble γ_s are considered as normal random variables with means μ_{γ_c}, μ_{γ_s}, and standard deviations σ_{γ_c}, σ_{γ_s}, respectively.

4. In an attempt to consider all the sources of uncertainty, the uncertainties of the formulas used in the computations have to be examined. In any case a calibration factor is applied to the result of the formula providing the true value.

(a) The Goda formulae for pulsating wave forces are biased in order to provide a safe relation (see Van der Meer [33] and Oumeraci *et al.* [26]). The uncertainty is taken into account using the calibration factors A_g, B_g, M_{A_g}, M_{B_g} and S_g affecting horizontal forces (F_h), uplift forces (F_v), horizontal moments (M_h), uplift moments (M_v) and seepage horizontal forces, respectively.

(b) The reliability of the overtopping prediction formula (56) can be expressed assuming a normal distribution for the random variable b_o, thus $b_o \sim N(\mu_{b_o}, \sigma_{b_o})$ (see Franco and Franco [31]). Note that the coefficient a in (56) is considered deterministic.

(c) The berm stability function ϕ_e in (58) uncertainty is considered due to the normal random coefficient $C_{ar} \sim N(\mu_{C_{ar}}, \sigma^2_{C_{ar}})$.

5. To consider model uncertainties for the limit state equations model factors equivalent to global safety factors are considered. These will be random parameters F_m (m refers to failure mode) log-normally distributed with expected values μ_{F_m} and coefficients of variation v_{F_m}.

Table 1: Statistical model and random model parameters $\boldsymbol{\kappa}$.

i	X_i	Meaning	Mean (μ)	Parameters	Distrib.
1	b	Leeward berm width (m)	$\bar{b}$	$v_b = 0.1$	Normal
2	B_m	Seaward berm width (m)	$\bar{B}_m$	$v_{B_m} = 0.1$	Normal
3	e	Armor protection thickness (m)	$\bar{e}$	$v_e = 0.1$	Normal
4	h_n	Rubble core height (m)	$\bar{h}_n$	$v_{h_n} = 0.1$	Normal
5	α_ℓ	Leeward slope angle (rad)	$\bar{\alpha}_\ell$	$v_{\alpha_\ell} = 0.1$	Normal
6	α_s	Seaward slope angle (rad)	$\bar{\alpha}_s$	$v_{\alpha_s} = 0.1$	Normal
7	$H_{s_{max}}$	Maximum significant wave height (m)		$\kappa_s = -0.1197$ $\delta_s = 0.446$ $\lambda_s = 3$	Pareto
		H_{max} obtained from linear regression of $H_{max}\|H_{s_{max}}$		$a_r = -0.641855$ $b_r = 1.92856$	
8	H_{max}	Residual between the maximum wave height (m) & the one obtained from above		$\kappa_w = 0.172482$ $\delta_w = 0.470151$ $\lambda_w = -0.201646$	Weibull
		$T_{z_{max}}$ obtained from linear regression of $T_{z_{max}}\|H_{max}, H_{s_{max}}$		$a_t = 5.66953$ $b_t = 3.5765$ $c_t = -1.35536$	
9	$T_{z_{max}}$	H_{max} wave period (seg)		$\sigma_{T_{z_{max}}} = 1.6128$	Normal
10	h_1	Tidal water level (m)		$h_{lo} = 20$ $t_r = 5$	Cosine
11	h_2	Meteorological water level (m)	0.02414	$\sigma_{h_2} = 0.11597$	Normal
12	θ_w	Incident wave angle (rad)	0.0	$\sigma_{\theta_w} = \pi/18$	Normal
13	A	Random wave breaking coefficient	0.15	$\sigma_A = 0.015$	Normal
14	μ_c	Friction factor caisson-rubble	0.636	$\sigma_{\mu_c} = 0.0954$	LN
15	μ_s	Friction factor rubble-rock	0.5	$v_{\mu_s} = 0.1$	LN
16	ϕ_r	Rubble friction factor (rad)	0.601	$v_{\phi_r} = 0.1$	Normal
17	γ_c	Average density of caisson (kN/m^3)	22.3	$\sigma_{\gamma_c} = 0.11$	Normal
18	γ_s	Rubble unit weight (kN/m^3)	21	$\sigma_{\gamma_s} = 0.11$	Normal
19	A_g	F_h model uncertainty	0.9	$\sigma_{A_g} = 0.2$	LN
20	B_g	F_v model uncertainty	0.77	$\sigma_{B_g} = 0.2$	LN
21	M_{A_g}	M_h model uncertainty	0.72	$\sigma_{M_{A_g}} = 0.37$	LN
22	M_{B_g}	F_v model uncertainty	0.72	$\sigma_{M_{B_g}} = 0.34$	LN
23	S_g	Seepage model uncertainty	0.65	$\sigma_{S_g} = 0.30$	LN
24	b_o	Overtopping model uncertainty	3	$\sigma_{b_o} = 0.26$	Normal
25	C_{ar}	Stability function uncertainty	1	$\sigma_{C_{ar}} = 0.1$	Normal
	F_m	$m = s, t, b, c, d, sea$	1	$v_{F_m} = 0.2$	LN
32	F_a	Armor failure uncertainty, $m = a$	1	$v_{F_a} = 0.1$	LN
33	F_o	Overtopping failure uncertainty, $m = o$	1	$v_{F_o} = 0.1$	LN

Note, for example, that in the overtopping failure, F_o takes into account the uncertainty of the critical structural safety discharge q_0.

All these assumptions and the numeric values used in the example are listed in Table 1.
Dependence assumptions The group of random variables $\{H_{s_{max}}, H_{max}, T_{z_{max}}\}$ are assumed to be dependent with the marginal and conditional distributions given above. For the sake of simplicity, the tidal water level is assumed to be independent of the remaining variables, and the same assumption is used for the meteorological tide; note however that this hypothesis is not really valid because it is dependent on $H_{s_{max}}$. The same would be applicable if storm surge effect in

Table 2: Fixed deterministic parameters used in the numerical example.

i	X_i	Meaning	Value (μ)	Units
1	a	Structure shape coefficient	0.082	$--$
2	$C_{a\ell}$	Armor layer construction cost per unit volume	70	$\$/m^3$
3	C_c	Sand filled caisson construction cost per unit volume	123	$\$/m^3$
4	C_{co}	Rubble core construction cost per unit volume	2.4	$\$/m^3$
5	D	Lifetime of the breakwater	50	$years$
6	f	Inflation rate	0.04	$--$
7	q_0	Maximum allowable mean overtopping discharge for structural damage	0.2	$m^3/s/m.l.$
8	r	Interest rate	0.0525	$--$
9	r_{p_s}	Sliding repair percentage	0.05	$--$
	r_{p_m}	$m = \{t, b, c, d, sea\}$ repair percentage	0.2	$--$
15	r_{p_o}	Overtopping repair percentage, $m = o$	0.01	$--$
16	r_{p_a}	Armor instability repair percentage, $m = a$	0.05	$--$
17	r_{st}	Mean number of storms per year	45.3427	$storms/year$
18	w_o	Caisson parapet width	2	m
19	γ_w	Water unit weight	10.35	kN/m^3
20	$\tan\theta_b$	Mean angle tangent of the sea bottom	1/50	$--$

shallow waters were considered. The remaining variables will be considered independent in this paper.

6 Numerical example

The proposed method has been implemented in GAMS (General Algebraic Modelling System) (see Castillo, Conejo, Pedregal, García and Alguacil [34]). GAMS is a software system especially designed for solving optimization problems (linear, non-linear, integer and mixed integer) of small to very large size. All the examples have been solved using the generalized reduced gradient method (for more details see VanderPlaats [35] or Bazaraa, Jarvis and Sherali [36]) that has shown good convergence properties including constraints to the variables. Of course, other optimization programs such as AIMMS, AMPL, LINDO, MPL or the Matlab Optimization Toolbox, can be used instead.

To illustrate the method, the automatic optimal design of a composite breakwater with the nominal values, statistical and cost parameters in Tables 1 and 2, respectively, has been performed.

The convergence of the process is attained after 46 iterations with an error tolerance lower than 10^{-5} as it is shown in Figure 4, but a reasonable solution is obtained after 26 iterations (error lower than 10^{-3}). It is worth mentioning that the convergence behavior is very good even using a starting design far away from the optimal. The final values of the optimized design variables $\bar{\boldsymbol{d}}$ are shown in Figure 5, which corresponds to a construction cost (C_0), a repair cost (C_r), and a total expected cost ($E[cost]$) of 74488.8, 28112.8 and 102601.6 \$, respectively. The optimal failure rates are:

$$r_s = 0.01365;\ r_t = 0.00098;\ r_b = 0.01903;\ r_c = 0.00545;$$
$$r_d = 0.00236;\ r_{sea} = 0.000;\ r_o = 0.11765;\ r_a = 0.05326.$$

Analysis of results The following conclusions can be drawn from the analysis of the results:

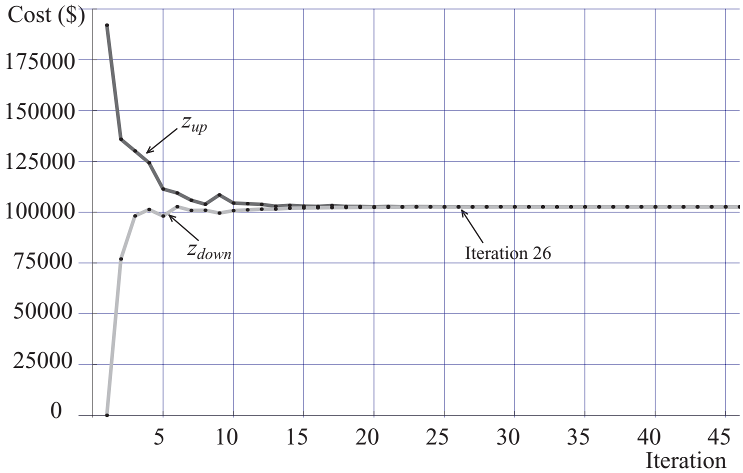

Figure 4: Graphical illustration of the reconstruction of the expected cost function using Benders cuts.

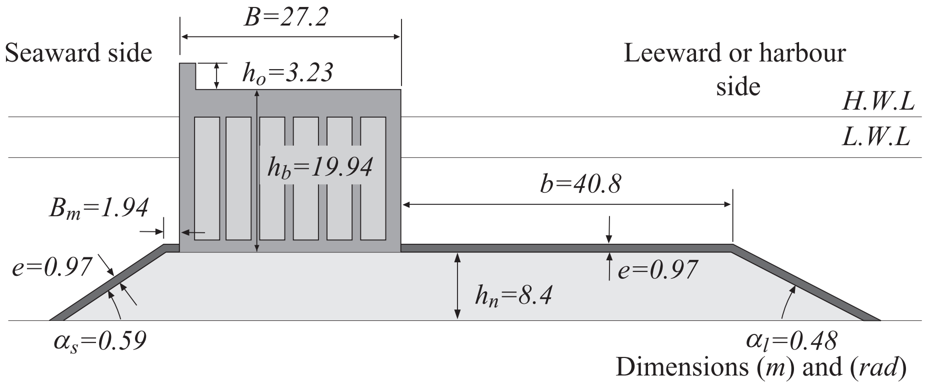

Figure 5: Scaled graphical illustration of the optimal composite breakwater.

1. The proposed method leads to the solution of the breakwater design, showing a good behavior. Note that in the computational example we have used 9 design variables, 33 statistical variables and 8 failure modes.

2. The optimal safety requirements show that the yearly overturning failure rate is very low because this failure mode is really dominated by the bearing capacity in rubble (rotation failure b). The yearly overtopping failure rate is bigger because consequences of failure are less important.

3. The cost sensitivities with respect to the cost of materials and some parameters of the model ($\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$) are given in Table 3. It allows one to know how much a small change in a single design factor value changes the optimal expected cost per running meter of the composite breakwater. This information is extremely useful during the construction process to control the cost, and for analyzing how the changes in the yearly failure rates required by the codes influence the total cost of maritime works. For example, a change of one unit in the cost of concrete C_c leads to a cost increase of 92982.9 \$ (see the corresponding entry in Table

Table 3: Some sensitivities of the total expected cost with respect $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$ parameters.

$\tilde{\eta}_i$	$\frac{\partial E[cost]}{\partial \tilde{\eta}_i}\lvert\tilde{\eta}_i\rvert$ (\$)	$\tilde{\eta}_i$	$\frac{\partial E[cost]}{\partial \tilde{\eta}_i}\lvert\tilde{\eta}_i\rvert$ (\$)	κ_i	$\frac{\partial E[cost]}{\partial \kappa_i}\lvert\kappa_i\rvert$ (\$)
μ_{ϕ_r}	-47011.8	C_{al}	7291.0	δ_S	58157.8
μ_{γ_c}	-2245856.2	C_c	92982.9	λ_S	251740.3
μ_{γ_s}	-2085151.2	D	20546.9	b_r	259900.9
μ_{A_g}	46821.3	r	-32290.2	a_T	52869.0
μ_{b_o}	-50156.5	f	24897.7	b_T	194463.0
μ_{F_b}	10867.2	r_{cb}	10606.6	c_T	-51501.4
μ_{F_c}	14465.8	r_{st}	28111.8	h_{lo}	66980.3
μ_{F_a}	10376.1	γ_w	2069737.2	t_r	55287.6

Table 4: Sensitivities of the the yearly failure rates r_m with respect the design variables ($\bar{\boldsymbol{d}}$).

$\bar{d}_i$	$\frac{\partial r_s}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_t}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_b}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_c}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_d}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_{sea}}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_o}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)	$\frac{\partial r_a}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$)
b	-	-	-	-0.009748	-0.004519	-	-	-
B	-0.051304	-0.009356	-0.108691	-0.023888	-0.009114	-0.000019	-	-
B_m	-	-	-	-	-	-0.000001	-	0.149561
e	0.002167	0.000140	0.003124	-0.000419	-0.000140	-	-	-0.211066
h_b	-0.060404	-0.004051	-0.040577	-0.016804	-0.005459	-	-1.772274	-
h_n	-0.003819	-0.001590	-0.000931	-0.000534	-0.004979	0.000022	-0.703502	0.112582
h_o	0.003984	0.000736	0.006557	0.002391	0.001250	-	-0.286670	-
α_ℓ	-	-	-	0.002363	0.001172	-	-	-
α_s	-	-	-	-	-	0.000013	-	-

3). Similarly, while an increase in the unit weight of the rubble mound γ_s decreases the cost (-2085151.2 \$), both the tidal range (t_r) and the zero port (h_{lo}) increase the cost by 55287.6 and 66980.3 \$ per relative unit increase, respectively.

4. The sensitivities of the yearly failure rates with respect to the optimized design variables $\bar{\boldsymbol{d}}$ are given in table 4. As an example, the influence of the freeboard on the verification equation for overtopping (57) and, therefore, on the corresponding yearly failure rate will be analyzed. This equation shows how this failure occurrence depends only on h_c, q_0, a, b_o and $H_{s_{max}}$. Note that increasing only the freeboard will lead to a safer structure. The freeboard is defined as $h_c = h_n + h_b + h_o - h$ with $h = h_1 + h_2$ and $h_{lo} \leq h_1 \leq h_{lo} + t_r$. Any increase on variables related to breakwater heights h_n, h_b, h_o will generate a decrease of yearly failure rate for overtopping due to the fact that all of them appear in the freeboard definition with positive sign.

7 Conclusions

The methodology presented in this paper provides a rational and systematic procedure for automatic and optimal design of engineering works. The engineer is capable of obtaining optimal yearly failure rates for the different modes of failure, so that the choice of the safety level for which the structure has to be designed taking into account the different consequences of a complete of partial failure depending on the structure and the environment is carried out. In addition, a sensitivity analysis can be easily performed by transforming the input parameters into auxiliary variables, which are set to their associated actual values. The provided example illustrates how this procedure can be applied and proves that it is practical and useful.

Some additional advantages of the proposed method are:

1. The method allows an easy connection with optimization frameworks.
2. The responsibility for iterative methods is given to the optimization software.
3. The reliability analysis takes full advantage of the optimization packages, which allows the solution of huge problems without the need of being an expert in optimization techniques.
4. Sensitivity values with respect to the target reliability levels are given, without additional cost, by the values of the dual problem.
5. It can be applied to different types of problems such as linear, non-linear, mixed-integer problems. The designer needs just to choose the adequate optimization algorithm.

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References

[1] M. A. Losada. Recent development in the design of mound breakwaters. In J.B. Herbich, editor, *Chapter 21 in Handbook of Coastal and Ocean Engineering*, volume I. Gulf Publishing, 1990.

[2] Puertos del Estado. Rom 0.0, Procedimiento general y bases de cálculo en el proyecto de obras marítimas y portuarias. Technical report, Puertos del Estado, Madrid, España, Noviembre 2001. pp 245.

[3] J. Galambos. *The Asymptotic Theory of Extreme Order Statistics.* Robert E. Krieger Publishing Company, Malabar, Florida, 1987.

[4] E. Castillo. *Extreme Value Theory in Engineering.* Academic Press, New York, 1988.

[5] C. Castillo, M. Losada, R. Mínguez, and E. Castillo. Técnicas de optimización aplicadas al diseño de obras marítimas. In *Procedimiento Metodológico Participativo para la Canalización, Recogida y Difusión de Estudios y Análisis Técnico-Científicos sobre los Documentos del Programa ROM, EROM 00*, EROM. Puertos del Estado, Ministerio de Fomento, 2003.

[6] C. Castillo, M. A. Losada, E. Castillo, and R. Mínguez. Técnicas de optimización aplicadas al diseño de obras marítimas. In *VII Jornadas de Ingeniería de Costas y Puertos*, pages 207–209. Fundaciń para el fomento de la ingeniería del agua, Almería, España, 2003.

[7] E. Castillo, R. Mínguez, A. Ruíz-Terán, and A. Fernández-Canteli. Design and sensitivity analysis using the probability-safety-factor method. An application to retaining walls. *Structural Safety*, 26(2):159–179, 2003.

[8] E. Castillo, A. Conejo, R. Mínguez, and C. Castillo. An alternative approach for addressing the failure probability-safety factor method with sensitivity analysis. *Reliability Engineering and System Safety*, 82(2):207–216, 2003.

[9] E. Castillo, M. Losada, R. Mínguez, C. Castillo, and A. Baquerizo. An optimal engineering design method that combines safety factors and failure probabilities: Application to rubble-mound breakwaters. *Journal of Waterways, Ports, Coastal and Ocean Engineering, ASCE*, 130(2):77–88, 2004.

[10] D. Van Dantzig. Economic decision problems for flood prevention. *Econometrika*, 24:276–287, 1956.

[11] H. G. Voortman, H. K. T. Kuijper, and J. K. Vrijling. Economic optimal design of vertical breakwaters. In *Proceedings of the 26th International Coastal Engineering Conference (ICCE)*, volume 2, pages 2124–2137, Copenhagen, Denmark, 1998. ASCE.

[12] I. Enevoldsen. *Reliability-based structural optimization.* PhD thesis, University of Aalborg, Aalborg, Denmark, 1991.

[13] I. Enevoldsen and J. D. Sorensen. Reliability-based optimization of series systems of parallel systems. *Journal of Structural Engineering*, 119(4):1069–1084, 1993.

[14] I. Enevoldsen and J. D. Sorensen. Reliability-based optimization in structural engineering. *Structural Safety*, 15:169–196, 1994.

[15] J. D. Sorensen, H.F. Burcharth, and E. Christiani. Reliability analysis and optimal design of monolithic vertical wall breakwaters. In *Proceedings 6th IFIP WG7.5 Working Conf. on Reliability and Optimization of Structural Systems*, Assissi, Italy, 1994. Chapman & Hall.

[16] H. D. Madsen, S. Krenk, and N. C. Lind. *Methods of structural safety.* Prentice Hall, Inc., Englewood Cliffs, New York, second edition, 1986.

[17] O. Ditlevsen and H. O. Madsen. *Structural reliability methods.* Wiley, Chichester, New York, 1996.

[18] R. E. Melchers. *Structural reliability analysis and prediction.* John Wiley & Sons, New York, second edition, 1999.

[19] J. F. Benders. Partitioning procedures for solving mixed-variable programming problems. *Numerische Mathematik*, 4:238–252, 1962.

[20] A. M Geoffrion. Generalized benders decomposition. *JOTA*, 10(4):237–260, 1972.

[21] R. Mínguez, E. Castillo, and A. S. Hadi. Solving the inverse reliability problem using decomposition techniques. *Structural Safety, ASCE*, 27:1–23, 2005.

[22] I. Enevoldsen. Sensitivity analysis of reliability-based optimal solutions. *Journal of Engineering Mechanics*, 120(1):198–205, 1994.

[23] E. Christiani, H. F. Burchart, and J. D. Sorensen. Reliability based optimal design of vertical breakwaters modelled as a series system of failure. In *Proceedings of the 25th International Coastal Engineering Conference (ICCE)*, volume 2, pages 1589–1602, Orlando, Florida (USA), 1996. ASCE.

[24] H. F. Burcharth and J. D. Sorensen. Design of vertical wall caisson breakwaters using partial safety factors. In *Proceedings of the 25th International Coastal Engineering*, volume 2, Copenhagen, Denmark, 1998.

[25] J. D. Sorensen and H. F. Burcharth. Reliability analysis of geotechnical failure modes for vertical wall breakwaters. *Computers and Geotechnics*, 26:225–245, 2000.

[26] H. Oumeraci, A. Kortenhaus, W. Allsop, M. de Groot, R. Crouch, H. Vrijling, and H. Voortman. *Probabilistic Design Tools for Vertical Breakwaters.* Balkema Publishers, New York, 2001.

[27] Working Group 28 PIANC. Breakwaters with vertical and inclined concrete walls. Report, Maritime Navigation Commission (MarCom), 2003.

[28] Y. Goda. New wave pressure formulae for composite breakwaters. In *Proceedings of the 14th International Coastal Engineering*, pages 1702–1720, Copenhagen, Denmark, 1974.

[29] S. Takahashi, K. Tanimoto, and S Shimosako. Experimental study of impulsive pressures on composite breakwaters. Technical Report 5, Port and Harbour Research Institute, 1992.

[30] Y. Goda. *Random Seas and Design of Maritime Structures.* Tokyo University Press, Tokyo, 1985.

[31] C. Franco and L. Franco. Overtopping formulas for caissons, breakwaters with nonbreaking 3D waves. *Journal of Waterway, Port, Coastal and Ocean Engineering*, 125(2):98–108, 1999.

[32] K. Tanimoto, T. Yagyu, and Y. Goda. Irregular wave tests for composite breakwater foundation. In *Proceedings of the 18th Conference on Coastal Engineering*, pages 2144–2163, Capetown, 1982.

[33] D.J. Van Der Meer, K. D'Angremond, and J. Juhl. Uncertainty on Goda formula. In *Proceedings of the 24th International Coastal Engineering*, Kobe, Japan, 1994.

[34] E. Castillo, A. Conejo, P. Pedregal, R. García, and N. Alguacil. *Building and Solving Mathematical Programming Models in Engineering and Science.* John Wiley & Sons Inc., New York, 2001. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts.

[35] G. N. Vanderplaats. *Numerical Optimization Techniques for Engineering Design.* McGraw-Hill, New York, 1984.

[36] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali. *Linear Programming and Network Flows.* John Wiley & Sons, New York, second edition, 1990.

A Appendix: Cost function

Consider the composite breakwater in Figure 2. To derive the cost function the following parts are considered:

Concrete volume: The caisson volume is:

$$V_c = Bh_b + w_o h_o. \tag{80}$$

Armor layer volume: The armor layer volume is:

$$V_{a\ell} = e[B_m + b + h_n(1/\sin\alpha_s + 1/\sin\alpha_\ell) + 0.5e(1/\tan\alpha_s + 1/\tan\alpha_\ell)]. \tag{81}$$

Core volume: The core volume is:

$$V_{co} = h_n(b + B + B_m - e(\tan(\alpha_s/2) + \tan(\alpha_\ell/2)) + h_n(1/\tan\alpha_s + 1/\tan\alpha_\ell)/2). \tag{82}$$

Then, the construction cost per unit length becomes:

$$C_0 = C_c V_c + C_{a\ell} V_{a\ell} + C_{co} V_{co}. \tag{83}$$