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Optimal Engineering Design via Benders' Decomposition

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Abstract The optimal engineering design problem consists in minimizing the expected total cost of a infrastructure or equipment, including construction and expected repair costs, the latter depending on the failure probabilities of each failure mode. The solution becomes complex because the evaluation of failure probabilities involves one optimization problem per failure mode. This paper formulates the optimal engineering design problem as a bilevel problem, i.e., an optimization problem constrained by a collection of other interrelated optimization problems. The structure of this bilevel problem is advantageously exploited using Benders' decomposition to develop and report an efficient algorithm to solve it. An advantage of this approach is that the design optimization and the reliability calculations are decoupled. The proposed algorithm is structurally simple and exhibits high computational efficiency. Its practical interest is demonstrated through a realistic but simple case study, a breakwater design example with two failure modes: overtopping and armor instability.

Keywords Benders decomposition · breakwater design · civil engineering examples · FORMs · optimal design

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1 Introduction

1.1 Motivation and objective

Optimal design problems entail a complex tradeoff: a moderate building cost implies a high maintenance and repair cost throughout the live of the equipment being designed, while a high building cost results in a moderate maintenance and repair cost. It is thus necessary to achieve a design, the optimal one, which results in minimum building plus maintenance and repair (total) cost.

Building costs are easily expressed as a function of the design variables of the equipment (or infrastructure) being designed. However, maintenance costs depend of the failure modes of the equipment, and their computation requires solving a set of involved and interrelated optimization problems.

The optimal design problem can thus be formulated as an optimization problem involving a complex objective function (building and maintenance costs) and a set of design and operational constraints. The objective function comprises two terms: the building cost, which is a function of the design variables, and the maintenance and repair costs, whose evaluation requires the solution of a number of interrelated optimization problems. Constraints express building and operational restrictions and depend on the design variables.

Thus, the problem to be addressed involves an objective function that embeds itself a collection of optimization problems, which is subject to a number of constraints. This problem structure leads naturally to a bilevel formulation: the term of the objective function involving the collection of optimization problems is substituted by a set of variables that are added to the objective function. Then, each of these variables is made equal to each one of the optimization problems constituting the aforementioned objective function term, and added to the original problem as a constraint. This results in an optimization problem subject itself to a collection of optimization problems, i.e., to a bilevel problem (see Colson et al (2005); Conejo et al (2006)).

1.2 Literature review

Optimization and engineering design has been an area of great interest for engineers and scientists. Design of structural elements entails an iterative process which usually requires practical experience. The designer selects the values of design variables and parameters and checks that safety and functionality constraints are satisfied. This process is repeated until a safe and cost effective structure is obtained. Optimization procedures are a natural way to free the engineer from the mentioned cumbersome iterative process, ensuring that the best possible solution is obtained.

This is the reason why optimal engineering design has been widely studied in the literature. Two main paradigms exist, "Deterministic Structural Optimization" (DSO), where the safety is accomplished using safety factors and fixed values of the random parameters of the model, and "Reliability-Based Structural Optimization" (RBSO), where the random character of the parameters involved is considered through probability density functions. Note that the main difficulty pertaining to RBSO is the definition of the failure probability, which is used to estimate maintenance and repair costs. Indeed, the failure probabilities are complicated to estimate and expressions for their gradients with respect to the design variables are not available. This situation makes standard nonlinear programming algorithms non appropriate for solving optimization problems involving failure probabilities. Within this context, several authors have proposed theoretical guidelines and heuristics for different optimization problems involving failure probabilities. For an exhaustive review see Frangopol (1995).

Optimal design problems can be solved using smooth response surfaces (see Gasser and Schuëller (1998)), which combined with standard nonlineal programming algorithms may result in numerically robust procedures. However, the quality of the approximation depends on the quality of the approximating surfaces. Other attempts use first order reliability methods (FORM, see Ditlevsen and Madsen (1996)), such as Enevoldsen and Sorensen (1994), which express the probability of failure in terms of the reliability index. However, Royset et al (2006) claim that the reliability index may not have continuos gradients with respect to the design variables, so convergence to an optimal solution is not guaranteed if gradient-based optimization algorithms are used. To avoid this shortcoming by eliminating the reliability index calculation, Madsen and Friis Hansen (1992) replace the probability of failure by the optimality conditions pertaining to the design point or point of maximum likelihood. However, this approach requires second-order derivatives and it may also lead to numerically ill-conditioned problems. Alternatively, Royset et al (2001, 2006) propose a decoupled approach where uncertainties can enter the objective function, the constraints, or both. It is based on a sequence of approximating design problems, which is constructed and solved using a semiinfinite optimization algorithm. Mínguez and Castillo (2009) propose also a decoupled approach based on decomposition techniques where the failure probabilities are calculated through first order reliability methods.

1.3 Aim and contribution

The aim of this paper is to provide a novel method based on Benders' decomposition to solve engineering design problems in a decoupled fashion, which allows taking advantage of recent state-of-the-art mathematical programming algorithms for: i) evaluating failure probabilities using FORM, ii) obtaining derivatives with respect to continuous variables, and iii) solving large scale problems (see Mínguez et al (2006)).

Under convexity assumptions, the typical manner of addressing a bilevel problem is to substitute the constraining lower-level problems by their corresponding KKT conditions, which results in a mathematical program with complementarity constraints (KKT constraints) so called MPCC. This MPCC is then attacked by conventional optimization algorithms.

Instead of using complementarity techniques that entail non-convexity and potential numerical ill-conditioning, we propose a Benders' decomposition algorithm (see Benders (1962); Lasdon (1970); Geoffrion (1972); Geoffrion and Graves (1974)) to attack the aforementioned optimal design problem. This algorithm avoids the use of KKT conditions as constraints, which results in robustness and computational efficiency. The complicating variables that allow advantageously decomposing the original problem are the auxiliary variables that allow moving the optimization problems in the objective function to the constraint set. In other words, the advantage of using Benders' Decomposition is to decompose the original problem into a set of subproblems of substantially reduced complexity. The price to be paid for such advantage is the need of an iterative algorithm.

Since Benders' decomposition has been widely analyzed in the existing literature, we dedicate space neither to repeat the advantages and shortcomings of this well known method, nor to explain when it is applicable. However, we indicate the contributions of this paper, which are fourfold:

1. To formulate a general optimal design problem as a bilevel problem.
2. To develop a Benders' decomposition algorithm tailored to solve the optimal design problem in item 1 above.
3. To implement dual variable techniques to obtain the partial derivatives required to implement Benders' decomposition.
4. To demonstrate the efficiency and robustness of the proposed Benders' approach solving a realistic case study.

1.4 Paper organization

The paper is organized as follows. In Section 2 we describe a general optimization problem whose objective function involves variables that require solving other optimization problems, and explain how it can be tackled by Benders' decomposition. In addition, we explain how the use of dual variables and an auxiliary optimization problem can help to obtain the partial derivatives required by Benders' decomposition. In Section 3 we illustrate the proposed method by using a simple example of breakwater design considering two failure modes: overtopping and armor instability. Finally, in Section 4 we provide some conclusions.

2 The Benders Decomposition

Consider the following problem structure:

$$\underset{\boldsymbol{x}}{\text{minimize}} \; f(\boldsymbol{x}) + \sum_{i=1}^{m} f_i(\boldsymbol{x}) \tag{1}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{3}$$

$$\left\{ \begin{array}{l} f_1(\boldsymbol{x}) = \underset{\boldsymbol{y}_1}{\text{minimum}} \; f_1(\boldsymbol{x}, \boldsymbol{y}_1) \\ \text{subject to} \\ \boldsymbol{a}_1(\boldsymbol{x}, \boldsymbol{y}_1) = \boldsymbol{0} \\ \boldsymbol{b}_1(\boldsymbol{x}, \boldsymbol{y}_1) \leq \boldsymbol{0}, \end{array} \right. \tag{4}$$

$$\vdots$$

$$\left\{ \begin{array}{l} f_m(\boldsymbol{x}) = \underset{\boldsymbol{y}_m}{\text{minimum}} \; f_m(\boldsymbol{x}, \boldsymbol{y}_m) \\ \text{subject to} \\ \boldsymbol{a}_m(\boldsymbol{x}, \boldsymbol{y}_m) = \boldsymbol{0} \\ \boldsymbol{b}_m(\boldsymbol{x}, \boldsymbol{y}_m) \leq \boldsymbol{0}, \end{array} \right. \tag{5}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y}_i \in \mathbb{R}^{s_i}$, $\forall i = 1, \ldots, m$, $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$, $f_i(\boldsymbol{x}, \boldsymbol{y}_i) : \mathbb{R}^n \times \mathbb{R}^{s_i} \to \mathbb{R}$, $\forall i = 1, \ldots, m$, $\boldsymbol{h}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^p$, $\boldsymbol{g}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^q$, $\boldsymbol{a}_i(\boldsymbol{x}, \boldsymbol{y}_i) : \mathbb{R}^n \times \mathbb{R}^{s_i} \to \mathbb{R}^{u_i}$, $\forall i = 1, \ldots, m$, and $\boldsymbol{b}_i(\boldsymbol{x}, \boldsymbol{y}_i) : \mathbb{R}^n \times \mathbb{R}^{s_i} \to \mathbb{R}^{v_i}$, $\forall i = 1, \ldots, m$.

The problem includes both equality and inequality constraints. The objective function is composed by functions which are the result of other optimization problems, $f(\boldsymbol{x})$ corresponds to the initial or construction costs and $f_i(\boldsymbol{x})$ represents the maintenance and repair costs for failure mode i. Variables $\boldsymbol{x}$ are complicating variables, i.e., variables that if fixed to given values render a decomposable problem dependent on $\boldsymbol{y}_i$, $\forall i = 1, \ldots, m$, which allows easily evaluating the objective function (1) by solving the m subproblems (4)-(5).

Problem (1)-(5) has the appropriate structure to apply the Benders decomposition advantageously. The aim of such decomposition is to reproduce the objective function (1) as a function solely of the complicating variables $\boldsymbol{x}$. If these variables are fixed to specific values using constraints of the form $\boldsymbol{x} = \boldsymbol{x}^{(\nu)}$, the objective function at iteration (ν) is evaluated solving the following problems:

$$f_i\left(\boldsymbol{x}^{(\nu)}\right) = \underset{\boldsymbol{y}_i}{\text{minimize}}\ f_i(\boldsymbol{x}, \boldsymbol{y}_i) \tag{6}$$

subject to

$$\boldsymbol{a}_i(\boldsymbol{x}, \boldsymbol{y}_i) = \mathbf{0} \tag{7}$$

$$\boldsymbol{b}_i(\boldsymbol{x}, \boldsymbol{y}_i) \leq \mathbf{0} \tag{8}$$

$$\boldsymbol{x} = \boldsymbol{x}^{(\nu)} : \boldsymbol{\lambda}_i^{(\nu)}, \tag{9}$$

for $i = 1, \dots, m$. The problems above are denominated subproblems, and their solutions provide values for the non complicating variables, $\boldsymbol{y}_i^{(\nu)}$, and the dual variable vector $\boldsymbol{\lambda}_i^{(\nu)}$, associated with those constraints that fix the complicating variables to given values. These dual variables supply information for building the Benders' cuts. An alternative way of evaluating the sensitivities of the objective function (6) with respect to $\boldsymbol{x}$ can be seen in Castillo et al (2006b). The resulting objective function of the original problem (1) is

$$f\left(\boldsymbol{x}^{(\nu)}\right) + \sum_{i=1}^{m} f_i\left(\boldsymbol{x}^{(\nu)}, \boldsymbol{y}_i^{(\nu)}\right)$$

which is an upper bound $z_{\text{up}}^{(\nu)}$ of the optimal objective function value because problems (6)-(9) are more constrained than the original one.

The information obtained solving the subproblem allows reproducing more and more accurately the original problem. Moreover, if the objective function in (1) is convex with respect to variables $\boldsymbol{x}$, the following problem approximates from below the original one:

$$\underset{\alpha, \boldsymbol{x}}{\text{Minimize}} \quad \alpha \tag{10}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0} \tag{11}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0} \tag{12}$$

$$\alpha \geq f\left(\boldsymbol{x}^{(\nu)}\right) + \sum_{i=1}^{m} f_i\left(\boldsymbol{x}^{(\nu)}, \boldsymbol{y}_i^{(\nu)}\right) + \sum_{k=1}^{n} \left(\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} + \sum_{j=1}^{m} \lambda_{jk}^{(\nu)}\right)\left(x_k - x_k^{(\nu)}\right). \tag{13}$$

Constraint (13) is the so-called Benders' cut and the problem (10)-(13) is denominated master problem. The optimal objective function value of this problem is a lower bound of the optimal objective function value of the original problem because problem (10)-(13) is a relaxation of the original problem. The solution of this master problem

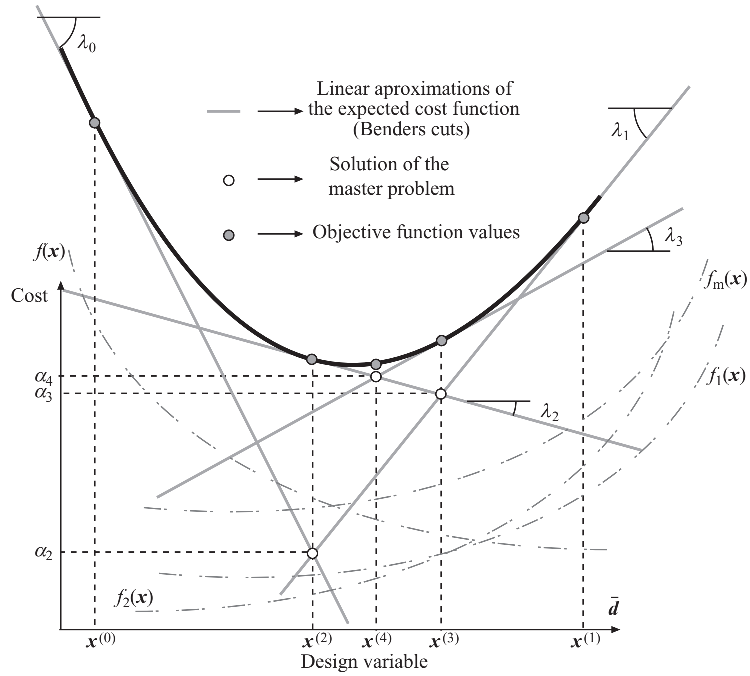

Fig. 1 Graphical illustration of how the objective function is approximated using Benders cuts.

provides new values for the complicating variables that are used for solving the subproblems again. Using the new information provided by those subproblems is possible to generate additional Benders cuts:

$$\alpha \geq f\left(\boldsymbol{x}^{(l)}\right) + \sum_{i=1}^{m} f_i\left(\boldsymbol{x}^{(l)}, \boldsymbol{y}_i^{(l)}\right) + \sum_{k=1}^{n}\left(\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} + \sum_{j=1}^{m} \lambda_{jk}^{(l)}\right)\left(x_k - x_k^{(l)}\right);\ l = 1, \ldots, \nu, \tag{14}$$

which allows us, using information from the previous ν iterations, formulating a more accurate master problem that provides new values of complicating variables.

For the one-dimensional case the derivatives on the Benders cuts correspond to the slopes of the approximating hyperplanes as illustrated in Figure 1.

The procedure continues until upper and lower bounds of the objective function optimal value are close enough.

Note from expression (14) that the partial derivatives $\dfrac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{\lambda}^{(\nu)^0}$ are required. These can be obtained analytically or by solving the following auxiliary optimization problem:

$$\underset{\boldsymbol{x}}{\text{minimize}}\ f(\boldsymbol{x}) \tag{15}$$

subject to

$$h(x) = 0 \quad (16)$$

$$g(x) \leq 0 \quad (17)$$

$$x = x^{(\nu)} : \lambda^{(\nu)^0}, \quad (18)$$

where $\lambda^{(\nu)^0}$ are the dual variables associated with the constraints (18). Note that problem (15)-(18) has a single solution given by constraint (18), $x = x^{(\nu)}$.

2.1 Algorithm

Benders' decomposition algorithm for solving problem (1)-(5) works as follows.

Input. A small tolerance ε to convergence control and initial guesses of the complicating variables x_0.

Output. The solution of problem (1)-(5)

– **Step 0: Initialization.** Set $\nu = 1$, $x^{(\nu)} = x_0$, $z_{\text{down}}^{(\nu)} = -\infty$.

  Step 1: Subproblem and auxiliary problem solutions. Solve the subproblems (6)-(9). The solution of these subproblems provide $y_i^{(\nu)}$, and $\lambda_i^{(\nu)}$, $i = 1, \ldots, m$. Update the objective function upper bound,

$$z_{\text{up}}^{(\nu)} = f\left(x^{(\nu)}\right) + \sum_{i=1}^{m} f_i\left(x^{(\nu)}, y_i^{(\nu)}\right).$$

  Solve auxiliary problem (15)-(18) for $x = x^{(\nu)}$ to obtain $\dfrac{\partial f(x^{(\nu)})}{\partial x^{(\nu)}}$.

– **Step 2: Convergence check.** If $||z_{\text{up}}^{(\nu)} - z_{\text{down}}^{(\nu)}|| \leq \varepsilon$, the solution with a level of accuracy ε of the objective function is

$$x^* = x^{(\nu)}$$
$$y_i^* = y_i^{(\nu)};\ \forall i = 1, \ldots, m.$$

  Otherwise, the algorithm continues with the next step.

– **Step 3: Master problem solution.** Update the iteration counter, $\nu \leftarrow \nu + 1$ and solve the following master problem:

$$\underset{\alpha, x}{\text{Minimize}} \quad \alpha \quad (19)$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{20}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{21}$$

$$\alpha \geq f\left(\boldsymbol{x}^{(l)}\right) + \sum_{i=1}^{m} f_i\left(\boldsymbol{x}^{(l)}, \boldsymbol{y}_i^{(l)}\right) + \sum_{k=1}^{n}\left(\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} + \sum_{j=1}^{m} \lambda_{jk}^{(l)}\right)\left(x_k - x_k^{(l)}\right);\ l = 1, \ldots, \nu - 1 \tag{22}$$

$$\alpha \geq \alpha_{\text{down}} \tag{23}$$

where α_{down} is a lower bound of α (heuristically obtained) to avoid an initial unbound solution. Alternatively and/or additionally, lower and upper bounds of the complicating variables $\boldsymbol{x}$ can be given.
Note that at every iteration a new constraint is added. The solution of the master problem provides $\boldsymbol{x}^{(\nu)}$, and $\alpha^{(\nu)}$.
Update the objective function lower bound, $z_{\text{down}}^{(\nu)} = \alpha^{(\nu)}$.
The algorithm continues in Step 1.

2.2 Benders Decomposition Convergence Analysis

The proposed algorithm provides the solution of the problem in a finite number of iterations if the objective function projected on the subspace of the complicating variables $\boldsymbol{x}$ is convex, otherwise, the procedure fails to converge (Geoffrion, 1972).

By definition, a function $F(\boldsymbol{x})$ is convex if and only if $F(\boldsymbol{y}) \geq F(\boldsymbol{x}) + \nabla F(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x})$ holds for all $\boldsymbol{x}$, $\boldsymbol{y} \in$ domain $F(\boldsymbol{x})$. This condition is equivalent to constraint (22), and if $l \to \infty$ it allows reproducing exactly the original objective function at the optimal solution neighborhood, which means that problems (1)-(5) and (19)-(22) are equivalent if and only if the function $F(\boldsymbol{x})$ is convex in the feasibility domain defined by (2)-(3). $F(\boldsymbol{x})$ is the objective function of problem (1)-(5) projected on the subspace of the complicating variables $\boldsymbol{x}$. In that case the original problem (1)-(5) and the master problem (19)-(22) are equivalent and converge to the same solution.

3 Example of application

In this section the proposed method is illustrated by its application to the reliability-based optimal design of a rubblemound breakwater (see Castillo et al (2004, 2006a); Mínguez et al (2006)). The purpose of such construction is to protect a harbor area from high waves during storms. The crest of the breakwater must be high enough to prevent the intrusion of sea water into the harbor by overtopping and the armor pieces

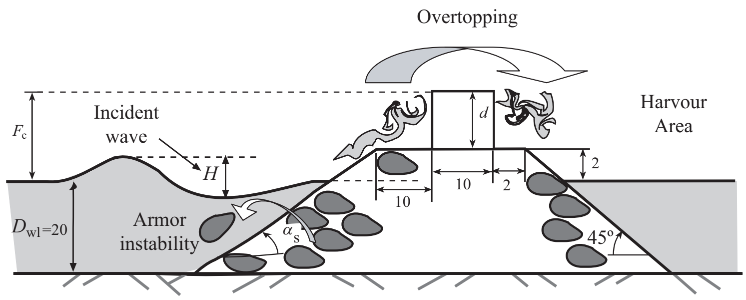

Fig. 2 Parameterized rubblemound breakwater used in the example.

large enough for the breakwater to be stable. For simplicity, only overtopping and armor instability failures are considered (see Figure 2).

In coastal engineering it is customary to consider a *given sea state*, which is the period of time d_{st} (usually one hour) in which the characteristics of the wave climate are assumed to be stationary, and are considered a reference for design. This given sea state is called *design* sea state, and it is defined as the sea state which happens on average once every R_p years, where R_p is the return period (usually 25, 50 or 100 years). During the design sea state, waves occur randomly and are assumed to be defined by their wave height H and wave period T, which follow a Rayleigh and a Bretscheider distribution, respectively. The Rayleigh distribution is defined through the parameter H_{s}, i.e., the significant wave height (approximately the average of the highest third of the wave heights in the design sea state), and the Bretscheider distribution is defined through the parameter $\bar{T}$, i.e., the wave mean period. There is also a random variable associated with the sea level, the storm surge η, which for this particular case is assumed to be normally distributed with zero mean and standard deviation σ_η. The storm surge explains the variations of the water level D_{wl} of the design sea state due to atmospheric conditions (high and low pressures).

For the sake of clarify, the following set of variables and/or parameters are defined: i) optimization design variables F_{c} (freeboard) and $\tan\alpha_{\text{s}}$ (rubblemound slope angle) whose values must be selected by the optimization procedure, ii) data selected by the designer $\{A_{\text{u}}, B_{\text{u}},\ D_{\text{wl}}, g, c_{\text{c}},\ c_{\text{a}}, H_{\text{s}}, \bar{T},\ d_{\text{st}}, \gamma_w,\ \gamma_s, \sigma_\eta,\ D, \gamma_{\text{ar}}, \gamma_{\text{ar}},\ W\}$, where A_{u} and B_{u} are given coefficients depending on the armor units to calculate the wave run-up, g is the gravity constant, constants c_{c} and c_{a} are the concrete and armor construction costs per unit volume, respectively, γ_w is the water unit weight, γ_s is the rubblemound unit weight, D is the breakwater lifetime, parameters γ_{ar} and γ_{ar} are the repair cost proportions with respect to the construction cost of the breakwater for overtopping and armor stability failures, respectively, and W is the weight of the armor pieces, iii) the random variables wave height H, period T and storm surge η, and finally, iv) the non-basic or auxiliary variable set $\{I_{\text{r}}, I_{\text{r}_0},\ v_{\text{a}}, v_{\text{c}},\ C_{\text{co}}, C_{\text{ov}}, C_{\text{ar}},\ L, d,\ P_{\text{f}_{\text{ov}}}, P_{\text{f}_{\text{ar}}},\ P_{\text{f}}^{\text{ov}}, P_{\text{f}}^{\text{ar}},$

$R_\mathrm{u}, \Phi_e, R\}$, where I_r is the Iribarren number, I_{r_0} is the Iribarren number for shallow water conditions, v_c and v_a are the concrete and armor volumes, respectively, C_co is the construction cost, C_ov and C_ar the expected overtopping and armor instability repair costs, L is the wave length, d is the caisson height, $P_{\mathrm{f}_\mathrm{ov}}$ and $P_{\mathrm{f}_\mathrm{ar}}$ are the probabilities of overtopping and armor instability for a single wave during the sea state, P_f^ov and P_f^ar are the probabilities of overtopping and armor instability during the *design* sea state, R_u is the wave run-up, i.e. the maximum excursion of water over the breakwater slope, Φ_e is the stability function and R is a dimensionless constant.

The construction cost C_co is given by

$$C_\mathrm{co} = c_\mathrm{c} v_\mathrm{c} + c_\mathrm{a} v_\mathrm{a}.$$

The expected repair costs due to overtopping and armor instability are considered to be given by

$$C_\mathrm{ov} = C_\mathrm{co} P_\mathrm{f}^\mathrm{ov} \gamma_\mathrm{ov} D \tag{24}$$

$$C_\mathrm{ar} = C_\mathrm{co} P_\mathrm{f}^\mathrm{ar} \gamma_\mathrm{ar} D. \tag{25}$$

The probabilities P_f^ov and P_f^ar are related to the corresponding probabilities for single waves $P_{\mathrm{f}_\mathrm{ov}}$ and $P_{\mathrm{f}_\mathrm{ar}}$ by the expressions

$$P_\mathrm{f}^\mathrm{ov} = 1 - \left(1 - P_{\mathrm{f}_\mathrm{ov}}\right)^{(d_\mathrm{st}/\bar{T})} \tag{26}$$

$$P_\mathrm{f}^\mathrm{ar} = 1 - \left(1 - P_{\mathrm{f}_\mathrm{ar}}\right)^{(d_\mathrm{st}/\bar{T})}, \tag{27}$$

which are obtained assuming that singles waves during a sea state are independent. Note that (26)-(27) allow the calculation of the probability of failure for a sea state considering that a failure occurs if any of the single waves during the sea state produce a failure (series system).

With the above approximation, overtopping failure occurs whenever the wave run-up R_u exceeds the freeboard F_c, i.e., if $F_\mathrm{c} - R_\mathrm{u} < 0$. Similarly, the armor instability failure occurs whenever $\gamma_w R \Phi_e H^3$ exceeds the weight of the armor block W, i. e., if $W \leq \gamma_w R \Phi_e H^3$, where the dimensionless constant R is

$$R = \frac{\gamma_s}{\gamma_w \left(\dfrac{\gamma_s}{\gamma_w} - 1\right)^3}.$$

Under all these considerations, for a rubblemound breakwater of slope $\tan\alpha_\mathrm{s}$ and freeboard F_c (see Figure 2), the reliability based design problem (1)-(5) consists of:

$$\underset{F_\mathrm{c}, \tan\alpha_\mathrm{s}}{\text{Minimize}} \; C_\mathrm{to} = (c_\mathrm{c} v_\mathrm{c} + c_\mathrm{a} v_\mathrm{a}) \left[1 + (P_\mathrm{f}^\mathrm{ov} \gamma_\mathrm{ov} + P_\mathrm{f}^\mathrm{ar} \gamma_\mathrm{ar})\; D\right] \tag{28}$$

subject to

$$F_{\rm c} = 2+d \tag{29}$$

$$v_{\rm c} = 10d \tag{30}$$

$$v_{\rm a} = \frac{1}{2}(D_{\rm wl}+2)\left(46+D_{\rm wl}+\frac{D_{\rm wl}+2}{\tan\alpha_{\rm s}}\right) \tag{31}$$

$$P_{\rm f_{ov}} = \Phi\left(-\beta^{\rm ov}\right) \tag{32}$$

$$P_{\rm f}^{\rm ov} = 1-\left(1-P_{\rm f_{ov}}\right)^{(d_{\rm st}/\bar{T})} \tag{33}$$

$$P_{\rm f_{ar}} = \Phi\left(-\beta^{\rm ar}\right) \tag{34}$$

$$P_{\rm f}^{\rm ar} = 1-\left(1-P_{\rm f_{ar}}\right)^{(d_{\rm st}/\bar{T})} \tag{35}$$

$$8 \le F_{\rm c} \le 15 \tag{36}$$

$$1/4 \le \tan\alpha_{\rm s} \le 1/2, \tag{37}$$

$$\left\{\begin{array}{ll} \beta^{\rm ov} = \underset{H,T,\eta}{\text{minimum}} \sqrt{z_1^2+z_2^2+z_3^2} & \text{subject to} \\ \frac{R_{\rm u}}{H} & = A_{\rm u}\left(1-e^{B_{\rm u}I_{\rm r}}\right) \\ I_{\rm r} & = \tan\alpha_{\rm s}/\sqrt{H/L} \\ L & = T\sqrt{g(D_{wl}+\eta)} \\ \Phi(z_1) & = 1-e^{-2(H/H_{\rm s})^2} \\ \Phi(z_2) & = 1-e^{-0.675(T/\bar{T})^4} \\ z_3 & = \eta/\sigma_\eta \\ F_{\rm c} & = R_{\rm u}, \end{array}\right. \tag{38}$$

$$\left\{\begin{array}{ll} \beta^{\rm ar} = \underset{H,T,\eta}{\text{minimum}} \sqrt{z_1^2+z_2^2+z_3^2} & \text{subject to} \\ A_r & = 0.2566-0.177/\tan\alpha_s+0.034/(\tan\alpha_s)^2 \\ B_r & = -0.0201-0.4123/\tan\alpha_s+0.055/(\tan\alpha_s)^2 \\ I_{\rm r_0} & = 2.656\tan\alpha_s \\ \Phi_e & = A_r(I_{\rm r}-I_{\rm r_0})\exp\left[b_r(I_{\rm r}-I_{\rm r_0})\right] \\ I_{\rm r} & \ge I_{\rm r_0} \\ I_{\rm r} & = \tan\alpha_{\rm s}/\sqrt{H/L} \\ \left(\frac{2\pi}{T}\right)^2 & = g\frac{2\pi}{L}\tanh\frac{2\pi(D_{\rm wl}+\eta)}{L} \\ \Phi(z_1) & = 1-e^{-2(H/H_{\rm s})^2} \\ \Phi(z_2) & = 1-e^{-0.675(T/\bar{T})^4} \\ z_3 & = \eta/\sigma_\eta \\ W & = \gamma_w R\Phi_e H^3, \end{array}\right. \tag{39}$$

where constraints (29) to (39) correspond to constraints (2) to (5) in problem (1)-(5). Note that the objective function (28) is written in terms of failure probabilities. Constraints (29)-(31) define geometric relations, constraints (32) to (35) stablish the relationships among different failure probabilities (single waves and sea state), and (36) and (37) force reasonable bounds for the freeboard and the rubblemound slope, respectively. Problems (38) and (39) are the subproblems and allow us calculating the reliability indexes associated with the overtopping and armor stability failures. Note that using First Order Reliability Methods (Ditlevsen and Madsen, 1996), the reliability index is related to the probability of failure through equalities (32) and (34), where $\Phi(\cdot)$ is the standard normal cumulative distribution function. The most likely values of the random variables H, T and η which produce the overtopping and armor stability failures, respectively, are also obtained.

Note that the Rosenblatt transformation (Rosenblatt (1952); Nataf (1962)) of the random variables H, T and η into independent standard normal random variables z_1, z_2 and z_3 is embedded in (38) and (39). The last constraints in (38) and (39) are the limit state equations forcing strict overtopping and armor stability failures, respectively.

For a given solution of the master problem, $\tan\alpha_s^*$ and F_{c}^*, the subproblems can be stated as follows:

$$\beta^{\mathrm{ov}} = \underset{H,T,\eta}{\text{Minimize}} \sqrt{z_1^2+z_2^2+z_3^2} \tag{40}$$

subject to

$$\frac{R_{\mathrm{u}}}{H} = A_{\mathrm{u}}\left(1-e^{B_{\mathrm{u}}I_{\mathrm{r}}}\right) \tag{41}$$
$$I_{\mathrm{r}} = \tan\alpha_{\mathrm{s}}/\sqrt{H/L} \tag{42}$$
$$L = T\sqrt{g(D_{wl}+\eta)} \tag{43}$$
$$\Phi(z_1) = 1-e^{-2(H/H_{\mathrm{s}})^2} \tag{44}$$
$$\Phi(z_2) = 1-e^{-0.675(T/\bar{T})^4} \tag{45}$$
$$z_3 = \eta/\sigma_\eta \tag{46}$$
$$F_{\mathrm{c}} = R_{\mathrm{u}} \tag{47}$$
$$\tan\alpha_s = \tan\alpha_s^* \;:\; \lambda_{\tan\alpha_s}^{\mathrm{ov}} \tag{48}$$
$$F_{\mathrm{c}} = F_{\mathrm{c}}^* \;:\; \lambda_{F_c}^{\mathrm{ov}}, \tag{49}$$

where $\lambda_{\tan\alpha_s}^{\mathrm{ov}}$ and $\lambda_{F_c}^{\mathrm{ov}}$ are the dual variables associated with constraints (48) and (49), respectively, and

$$\beta^{\mathrm{ar}} = \underset{H,T,\eta}{\text{minimum}} \sqrt{z_1^2+z_2^2+z_3^2} \tag{50}$$

subject to

$$A_r = 0.2566-0.177/\tan\alpha_s+0.034/(\tan\alpha_s)^2 \tag{51}$$
$$B_r = -0.0201-0.4123/\tan\alpha_s+0.055/(\tan\alpha_s)^2 \tag{52}$$
$$I_{\mathrm{r}_0} = 2.656\tan\alpha_s \tag{53}$$
$$\Phi_e = A_r(I_{\mathrm{r}}-I_{\mathrm{r}_0})\exp\left[b_r(I_{\mathrm{r}}-I_{\mathrm{r}_0})\right] \tag{54}$$
$$I_{\mathrm{r}} \geq I_{\mathrm{r}_0} \tag{55}$$
$$I_{\mathrm{r}} = \tan\alpha_{\mathrm{s}}/\sqrt{H/L} \tag{56}$$
$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{L}\tanh\frac{2\pi(D_{\mathrm{wl}}+\eta)}{L} \tag{57}$$

$$
\begin{aligned}
\Phi(z_1) &= 1 - e^{-2(H/H_s)^2} && (58)\\
\Phi(z_2) &= 1 - e^{-0.675(T/\bar{T})^4} && (59)\\
z_3 &= \eta/\sigma_\eta && (60)\\
W &= \gamma_w R \Phi_e H^3 && (61)\\
\tan\alpha_s &= \tan\alpha_s^* \ : \ \lambda^{\mathrm{ar}}_{\tan\alpha_s} && (62)\\
F_\mathrm{c} &= F_\mathrm{c}^* \ : \ \lambda^{\mathrm{ar}}_{F_c}, && (63)
\end{aligned}
$$

where $\lambda^{\mathrm{ar}}_{\tan\alpha_s}$ and $\lambda^{\mathrm{ar}}_{F_c}$ are the dual variables associated with constraints (62) and (63), respectively.

Finally, the master problem is

$$
\underset{\alpha, F_\mathrm{c}, \tan\alpha_s}{\text{Minimize}} \quad \alpha \tag{64}
$$

subject to

$$
\begin{aligned}
\alpha &\geq C_\mathrm{to}^{(l)} + \lambda^M_{F_\mathrm{c}}\left(F_\mathrm{c} - F_\mathrm{c}^{(l)}\right) + \lambda^M_{\tan\alpha_s}\left(\tan\alpha_s - \tan\alpha_s^{(l)}\right); \quad l = 1, \dots, \nu - 1 && (65)\\
8 &\leq F_\mathrm{c} \leq 15 && (66)\\
1/4 &\leq \tan\alpha_\mathrm{s} \leq 1/2 && (67)\\
\alpha &> 500, && (68)
\end{aligned}
$$

where the $\lambda^M_{F_\mathrm{c}} = \dfrac{\partial C_\mathrm{to}}{\partial F_\mathrm{c}}$ and $\lambda^M_{\tan\alpha_s} = \dfrac{\partial C_\mathrm{to}}{\partial \tan\alpha_s}$ are calculated as follows

$$
\begin{aligned}
\lambda^M_{F_\mathrm{c}} &= \lambda^+_{F_\mathrm{c}} \left[1 + (\gamma_\mathrm{ov} P_f^\mathrm{ov} + \gamma_\mathrm{ar} P_f^\mathrm{ar}) D\right] + C_\mathrm{co} D \left[\gamma_\mathrm{ov} \frac{\partial P_f^\mathrm{ov}}{\partial \beta^\mathrm{ov}} + \gamma_\mathrm{ar} \frac{\partial P_f^\mathrm{ar}}{\partial \beta^\mathrm{ar}}\right] && (69)\\
\lambda^M_{\tan\alpha_s} &= \lambda^+_{\tan\alpha_s} \left[1 + (\gamma_\mathrm{ov} P_f^\mathrm{ov} + \gamma_\mathrm{ar} P_f^\mathrm{ar}) D\right] + C_\mathrm{co} D \left[\gamma_\mathrm{ov} \frac{\partial P_f^\mathrm{ov}}{\partial \beta^\mathrm{ov}} + \gamma_\mathrm{ar} \frac{\partial P_f^\mathrm{ar}}{\partial \beta^\mathrm{ar}}\right], && (70)
\end{aligned}
$$

where $\lambda^+_{F_\mathrm{c}} = \dfrac{\partial C_\mathrm{co}}{\partial F_\mathrm{c}}$ and $\lambda^+_{\tan\alpha_s} = \dfrac{\partial C_\mathrm{co}}{\partial \tan\alpha_s}$, that can be calculated analytically or by solving the following auxiliary optimization problem

$$
\underset{F_\mathrm{c}, \tan\alpha_\mathrm{s}}{\text{Minimize}} \ C_\mathrm{co} = c_\mathrm{c} v_\mathrm{c} + c_\mathrm{a} v_\mathrm{a} \tag{71}
$$

subject to

$$
\begin{aligned}
F_\mathrm{c} &= 2 + d && (72)\\
v_\mathrm{c} &= 10d && (73)\\
v_\mathrm{a} &= \frac{1}{2}(D_\mathrm{wl} + 2)\left(46 + D_\mathrm{wl} + \frac{D_\mathrm{wl} + 2}{\tan\alpha_\mathrm{s}}\right) && (74)\\
F_\mathrm{c} &= F_\mathrm{c}^* \ : \ \lambda^+_{F_\mathrm{c}} && (75)\\
\tan\alpha_s &= \tan\alpha_s^* \ : \ \lambda^+_{\tan\alpha_s}, && (76)
\end{aligned}
$$

Table 1 Illustration of the iterative procedure. The design and final values are bolfaced.

Iterations	F_c (m)	$\tan\alpha_s$	C_{co} (\$)	C_{ov} (\$)	C_{ar} (\$)	C_{to} (\$)	$\boldsymbol{z}_{up}$ (\$)	$\boldsymbol{z}_{down}$ (\$)	error
1	9.000	0.360	7555.7	5905.5	122.4	13583.6	13583.6	500.0	1.0000
2	9.090	0.250	8319.7	154.3	19099.9	27574.0	13583.6	500.0	26.1672
3	10.264	0.360	8316.5	406.2	135.5	8858.1	8858.1	500.0	16.7163
4	15.000	0.361	11149.6	0.0	178.1	11327.7	8858.1	7280.2	0.3209
5	10.318	0.329	8499.5	134.4	364.6	8998.6	8858.1	8673.7	0.5528
6	10.555	0.348	8543.4	142.4	173.2	8859.0	8858.1	8699.1	0.0779
7	11.195	0.400	8709.6	144.3	232.4	9086.4	8858.1	8768.0	0.1878
8	10.753	0.375	8543.0	200.7	135.7	8879.5	8858.1	8778.4	0.1087
9	10.429	0.356	8429.2	251.2	143.3	8823.8	8823.8	8786.1	0.0835
10	10.015	0.334	8287.8	346.3	273.1	8907.2	8823.8	8806.0	0.1068
11	10.246	0.341	8391.4	245.5	210.0	8846.9	8823.8	8812.5	0.0429
12	10.199	0.348	8332.0	335.3	170.8	8838.1	8823.8	8816.2	0.0228
13	10.310	0.348	8398.3	257.4	171.6	8827.2	8823.8	8817.2	0.0112
14	10.308	0.352	8375.9	296.5	153.5	8825.9	8823.8	8820.1	0.0129
15	10.362	0.352	8410.4	256.9	155.7	8823.0	8823.0	8820.3	0.0065
16	10.494	0.353	8486.2	190.5	154.8	8831.5	8823.0	8821.8	0.0144
17	10.440	0.353	8449.8	223.1	151.5	8824.5	8823.0	8821.9	0.0076
18	10.410	0.354	8429.8	243.2	149.8	8822.8	8822.8	8822.0	0.0042
19	10.381	0.354	8410.1	264.7	148.2	8822.9	8822.8	8822.0	0.0041
20	10.385	0.353	8417.3	254.1	151.2	8822.6	8822.6	8822.3	0.0032
21	10.407	0.355	8423.5	252.3	147.1	8822.9	8822.6	8822.6	0.0063
22	10.392	0.354	8417.4	256.5	148.7	8822.7	8822.6	8822.6	0.0033
23	10.401	0.354	8424.0	249.2	149.4	8822.7	8822.6	8822.6	0.0015
24	10.394	0.354	8420.3	252.6	149.7	8822.6	8822.6	8822.6	0.0011
25	**10.388**	**0.354**	**8417.1**	**255.6**	**150.0**	**8822.6**	**8822.6**	**8822.6**	0.0009

where F_c^* and $\tan\alpha_s^*$ are the solution of the master problem and $\lambda^+_{F_c}$ and $\lambda^+_{\tan\alpha_s}$ are the dual variables associated with the last two constraints (75) and (76), respectively. The remainder derivatives $\frac{\partial P_f^{ov}}{\partial\beta^{ov}}$ and $\frac{\partial P_f^{ar}}{\partial\beta^{ar}}$ are calculated using the chain rule and the dual variables $\lambda^{ov}_{\tan\alpha_s}$, $\lambda^{ov}_{F_c}$, $\lambda^{ar}_{\tan\alpha_s}$ and $\lambda^{ar}_{F_c}$.

We assume the following values for the parameters involved:

$$\begin{array}{llll} A_u = 1.05; & B_u = -0.67; & D_{wl} = 20\ m; & g = 9.81\ m/s^2; \\ c_c = 60\ \$/m^3; & c_a = 2.4\ \$/m^3; & H_s = 6\ m; & \bar{T} = 12\ s; \\ d_{st} = 1\ h; & \gamma_w = 10.25\ kN/m^3; & \gamma_s = 26\ kN/m^3; & \sigma_\eta = 1\,m; \\ D = 25 \text{ years}; & \gamma_{ov} = 0.3; & \gamma_{ar} = 0.1; & W = 160\ kN, \end{array}$$

and $\varepsilon = 10^{-3}$ as the tolerance. Using the methods proposed in Section 2, the considered problem is solved using solver CONOPT (Drud, 1996) under the General Algebraic Modeling System (GAMS, www.gams.com), which is a high-level modeling system for mathematical programming and optimization.

The solution is provided in Table 1, where the evolution of the convergence process is illustrated. The method converges in 25 iterations within the admissible tolerance $\varepsilon = 0.001$.

The solution that minimizes the total expected cost (boldfaced in Table 1) is $C_{co}^* =$ \$8417.1, and the repair cost due to overtopping and armor stability failures are $C_{ov}^* =$

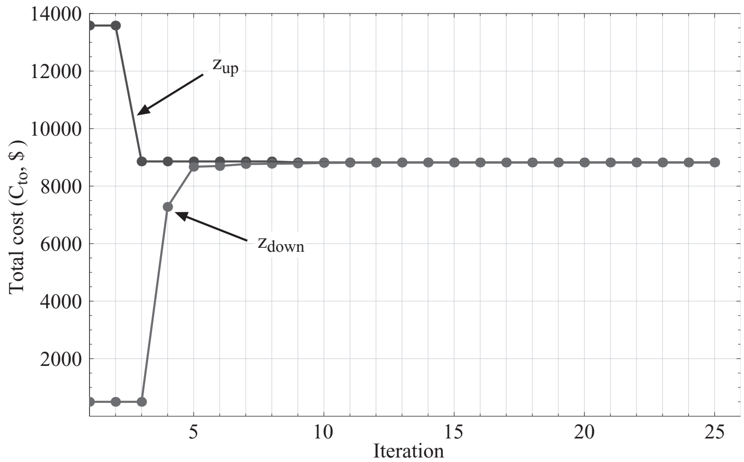

Fig. 3 Evolution of the objective function upper and lower bounds.

\$255.6 and $C^*_{\text{ar}} = \$150.0$, respectively, leading to a total cost of $C^*_{\text{to}} = \$8822.6$. The optimal design variable values $F^*_{\text{c}} = 10.388$ and $\tan\alpha^*_s = 0.354$, are also provided in Table 1.

In addition, the following optimal reliability values are obtained $\beta^{\text{ov}} = 4.199$, $P_{\text{f}_{\text{ov}}} = 1.35\times10^{-5}$, $P^{\text{ov}}_{\text{f}} = 4.05\times10^{-3}$ $\beta^{\text{ar}} = 4.068$, $P_{\text{f}_{\text{ar}}} = 2.38\times10^{-5}$ and $P^{\text{ar}}_{\text{f}} = 7.13\times10^{-3}$.

In Figure 3 the evolution of the lower and upper bound achieved through the iterative process up to 25 iterations is shown. Note that bounds converge monotonically to the optimal solution in 25 iterations within the admissible tolerance of $\varepsilon = 0.001$. The CPU time required using a processor clocking at 1.73 GHz and 2 GB of RAM is 3.7734 seconds.

Figure 4 shows the total cost function contour plot in terms of the two design variables F_{c} and $\tan\alpha_s$, where the darker the contour line is the lower the total cost is (see color scale on the right hand side of Figure 4). Black dots represent the master problem solutions at every iteration until the final solution (white dot) is achieved. The evolution of the master problem solutions is shown through the black line joining the solution points. Note that at iterations 2 and 4 the lower slope angle and the upper freeboard bounds become active, respectively.

Finally, Figure 5 provides a two-dimensional plot of α (the objective function) as a function of F_c and $\tan\alpha_s$ (the complicating variables), and two vertical cross sections for this two-dimensional plot (for $F_c = 10.45$ and $\tan\alpha_s = 0.35$). Note that the function is reasonably convex over the feasibility region. This justifies the functioning of the method.

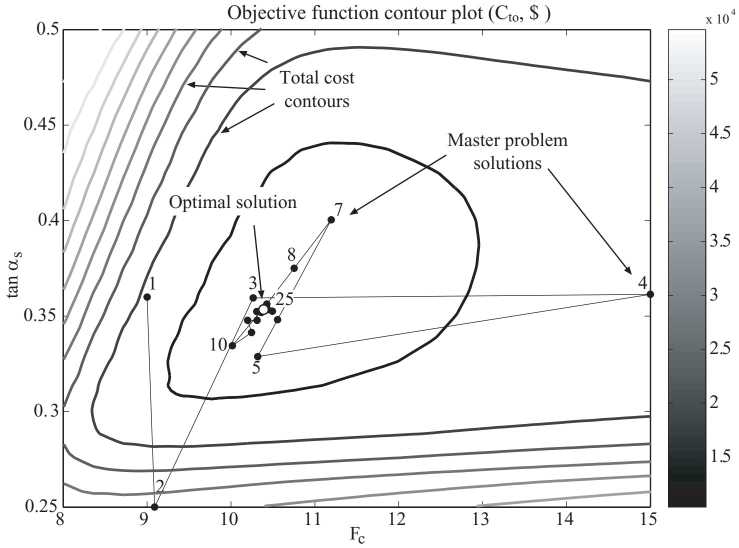

Fig. 4 Objective function contour plot as a function of the complicating variables (F_c, $\tan\alpha_s$). Darker contour lines correspond to lower total cost as color scale indicates.

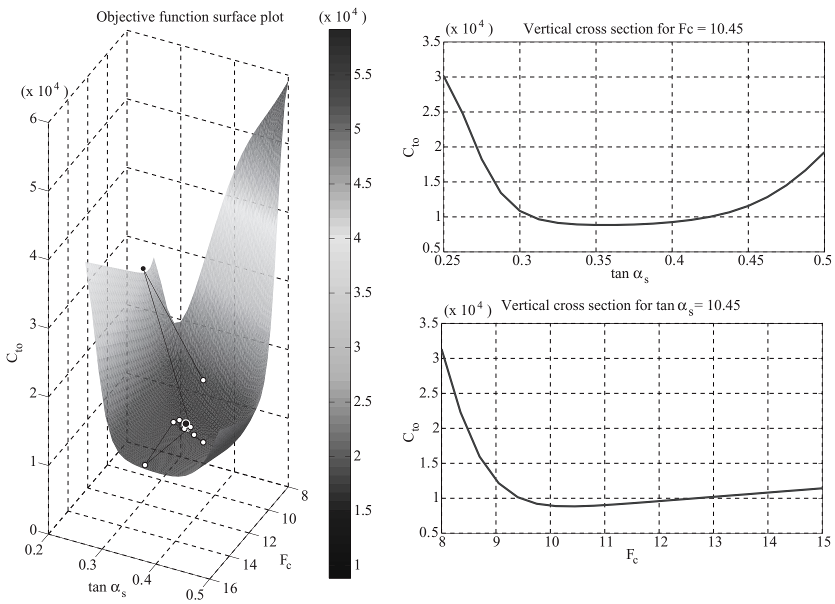

Fig. 5 Objective function α surface plot as a function of complicating variables F_c and $\tan\alpha_s$, and vertical cross sections of α for $F_c = 10.45$ and $\tan\alpha_s = 0.35$.

4 Conclusions

This paper uses Benders' decomposition to tackle optimal design problems in engineering. Considering the analysis reported in this paper, the following conclusions are in order:

1. The optimal design problem is naturally expressed as a bilevel problem with a structure exploitable via Benders' decomposition.
2. Benders' decomposition allows avoiding complementarity techniques, which are often cumbersome and hardly robust.
3. Under convexity assumption (of the objective function projected on the subspace of the complicating variables), the proposed Benders' decomposition algorithm is both efficient and robust, achieving the optimal solution in low computational time.
4. The use of dual variables and an auxiliary optimization problem allows computing efficiently the partial derivatives required to implement Benders' algorithm.
5. The use of first order reliability approximations for failure probability calculations and the decomposable structure of the solution strategy allows to take full advantage of recent state-of-the-art mathematical programming algorithms.

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