Reliability-based Optimization in Engineering using Decomposition Techniques and FORMS

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Abstract

In this paper a review of some decomposition techniques previously given by the authors to solve bi-level problems is presented within a unified formulation and a new variant is investigated. Different reliability-based optimization problems in engineering works are formulated and solved: (a) the failure-probability safety-factor problem that makes the compatibility of the classical approach, based on safety-factors, and the modern probability-based approach possible; (b) a modern reliability-based approach where design is based on minimizing initial/construction costs subject to failure-probability and safety-factor bounds for all failure modes; (c) minimizing the expected total cost of a structure, including maintenance and construction, which depend on the failure probabilities, and (d) a mixed model minimizing the expected total cost adding failure-probability and safety-factor bounds for all failure modes. In these four problems the objective consists of selecting the values of the design variables that minimize the corresponding cost functions subject to some reliability conditions together with geometric and code constraints. The solution becomes complex because the evaluation of failure probabilities using first order reliability methods (FORM) involves one optimization problem per failure mode, so that decomposition methods are used to solve the problem. The proposed methods use standard optimization frameworks to obtain the reliability indices and to solve the global problem within a decomposition scheme. An advantage of these approaches is that the optimization procedure and the reliability calculations are decoupled. In addition, a sensitivity analysis is performed using a method that consists of transforming the data parameters into artificial variables and using the dual associated variables. To illustrate the methods, a breakwater design example is used.

Key words: failure-probability safety-factor method, Benders' decomposition, sensitivity analysis, civil engineering examples, breakwater design.
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1 INTRODUCTION AND MOTIVATION

Design of structural elements in engineering is normally done by an iterative process, which usually requires practical experience. In each iteration, the engineer or the computer selects the values of design variables or parameters and checks that the safety and functionality constraints are satisfied. This process is repeated until a safe and cost reasonable structure is obtained.

Optimization procedures are a good solution to free the engineer from the above mentioned cumbersome iterative process, i.e., to automate the design process [1]. In this case, the values of the design variables are given by the optimization process and the engineer can fully concentrate on fixing the constraints, defining the objective function to be optimized, and analyzing the resulting structure.

Safety of structures is the most fundamental criterion for design. To this end, the engineer first identifies all failure modes of the work being designed and then establishes the safety constraints to be satisfied. To ensure satisfaction of the safety constraints, two approaches are normally used: (a) the classical safety-factor approach, and (b) the probability-based approach:

(1) *The classical approach* is based on safety-factors, which are used to guarantee the required safety of the structures to be designed.
    A classical design fixes the values of the safety-factors and chooses the values of the design variables to satisfy these safety conditions. The greater the damage associated with the failure mode, the greater the level of safety required for this mode. All the variables involved are assumed to be deterministic and initially random variables are fixed to particular quantiles, i.e, mean value or characteristic values.
(2) *The probability-based approach* works with probabilities of failure. Normally, a global probability of failure is used as the basic design criterion. However, working with failure probabilities is difficult because (a) it requires the definition of the joint probability of all variables involved, and (b) the evaluation of the failure-probability is not an easy task. The problem becomes even more difficult if several failure modes are analyzed, because the system failure region is the union of the different failure mode regions, and regions defined as unions are difficult to deal with because of their irregular and non-differentiable boundaries [2]. As an alternative design criterion, the probabilities of failure for the different modes can be

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used independently. Nevertheless, one may easily obtain upper and lower bounds for the system failure-probability [3].

A probability-based design checks that the selected design leads to failure probabilities below given upper bounds. Some or all the variables involved are assumed to be random.

In engineering design two main philosophies exist, "Deterministic Structural Optimization" (DSO), where the safety is accomplished using safety factors, and "Reliability-Based Structural Optimization" (RBSO), where the random character of the variables involved is considered through density functions. In the last few decades, there has been considerable research focusing on these topics, for example, Der Kiureghian [4], Geyskens [5], Sorensen and Faber [6], Neuenhofer and Zilch [7], Parkat al. [8], Zhang and Der Kiureghian [9], Polak et al. [10], Royset et al. [11], Enevoldsen [12], Enevoldsen and Sorensen [13]. For an exhaustive review see Frangopol [14]. In these works, different approaches are proposed, such as minimizing the cost subject to safety constraints using safety-factors, maximizing utility, and considering the expected cost during the lifetime of the structure, which allows maintenance and repair costs, etc. to be included.

In this paper, based on these approaches four bi-level reliability types of design problems are considered:

(1) *The failure-probability safety-factor method (FPSF).* Classic engineers criticize the probabilistic approach because of its sensitivity to statistical hypotheses, especially tail assumptions [15,16]. Similarly, probability-based engineers question classical designs because it is not clear how far their designs are from failure. To avoid the lack of agreement between defenders of both approaches, and to obtain a more reliable design the combined safety-factors and failure-probability constraints method has been proposed [17,18].
(2) *Design methods based on minimizing initial/construction costs.* Other authors minimize initial/construction costs subject to probability bounds for all failure modes, without considering the classical approach based on safety-factors [19,18,20].
(3) *Design methods based on total costs.* Some authors go further including in their reliability-based design problems the total cost (construction, maintenance and repairs) where the objective function itself depends on the probabilities of failure for each failure mode, so that these probabilities are also variables of the problem [13,21].
(4) *Design methods based on the mixed approach.* In this paper a new method combining the three previous approaches is considered, so that the most restrained safety conditions between different approaches prevail.

The main problem of all these approaches is that the failure probabilities needed to establish the corresponding bounds, or to get the value of the objective function if repair is included, are the solution of optimization problems themselves (using FORM). Since the failure-probability bounds cannot be directly imposed in the form of standard constraints, use of optimization programs is not straightforward, and special methods are needed. In particular, several decomposition iterative schemes to solve the problem [22] are proposed. In this way these problems are solved in a distributed solution, which allows us to solve large problems which cannot be dealt with using traditional optimization routines.

In this paper a review of some decomposition techniques previously given by the authors is presented and a new variant is investigated. To illustrate the methods, a completely new example of a breakwater is used. In particular, we wish to clarify that this paper is not a state of knowledge on decomposition techniques used in reliability problems.

In addition to requiring optimal solutions to problems, some interest is shown by people in knowing how sensitive the solutions to the assumed data values are. A sensitivity analysis provides excellent information on the extent to which a small change in the parameters or assumptions (data) modifies the resulting design (geometric dimensions, costs, reliabilities, etc.).

Because sensitivity analysis and decomposition techniques are very closely related, in this paper, in addition to presenting several decomposition techniques for solving different bi-level reliability-based optimization problems, tools to perform a sensitivity analysis are provided.

Note that these methods present some limitations with respect to other available methods existing in the literature: (a) they are limited to problems where the failure modes are considered independent, thought approximate bounds on the system probabilities could be added easily, and (b) it uses FORMS to obtain the failure probabilities. Note that second order reliability methods (SORM, [23–27], etc.) could be included, but it would make the formulation of the problem more complicated and fuzzier because curvature of the failure region must be obtained. The proposed methods are intended to facilitate the use of optimization frameworks when solving the RBO problem once constraints have been defined. On the other hand, the use of importance sampling (see [28] and [29]) methods to obtain the probabilities of failure is not an appropriate method at the design stage because the sensitivities of those probabilities of failure with respect the design variables are more difficult to obtain. It could, however, be an excellent method to check the effectiveness of the design solution and the individual failure mode bounds on the global probability of the design given by the approximate methods proposed.

The paper is structured as follows. In Section 2 the different reliability-based design models are introduced and the description of the decomposition techniques for solving the different models is shown. In Section 3 the technique for performing a sensitivity analysis is explained. In Section 4 an illustrative example is shown. Finally, Section 5 gives some conclusions.

2 RELIABILITY-BASED OPTIMIZATION

In this section we introduce some basic concepts that are needed to understand the subsequent material.

The design and reliability analysis of an engineering work involves a number of random variables $(X_1, \dots, X_n)$. These include geometric variables, material properties, loads, etc. They belong to an n-dimensional space, which can be divided into two regions, the safe S and the failure F regions:

$$\begin{aligned} S &\equiv \{(x_1, x_2, \dots, x_n) | g(x_1, x_2, \dots, x_n; F^0) \geq 0\}, \\ F &\equiv \{(x_1, x_2, \dots, x_n) | g(x_1, x_2, \dots, x_n; F^0) < 0\}, \end{aligned} \tag{1}$$

where $g(x_1, x_2, \dots, x_n; F^0)$ can be related to the non-dimensional ratio of two opposing magnitudes, such as stabilizing to overturning forces, strengths to ultimate stresses, etc. and $F^0 = 1$ is the global safety-factor. Since the constraint $g(x_1, x_2, \dots, x_n; F^0) = 0$ defines the limit state, to increase safety, the constant F^0 is normally replaced by a larger constant, $F^0 > 1$. If m different modes of failure are considered, the problem modifies to

$$S_i \equiv \{(x_1, x_2, \dots, x_n) | g_i(x_1, x_2, \dots, x_n; F_i^0) \geq 0\}, \tag{2}$$

where $i = 1, 2, \dots, m$.

One must distinguish between design values (those designed by the engineer), which in this paper are assumed to be the expectations ($E(X_i)$ or $\bar{x}_i$) or characteristic values ($\tilde{x}_i$), of the random variables $X_i : i = 1, 2, \cdots, n$, and actual values x_i (those existing in reality). Some of these expectations are chosen by the engineer or the design codes, and some are selected by the optimization procedure to be presented.

It is important and clarifying to classify the set of variables involved in an engineering design problem into the following four subsets:

- ***d***: *Optimization design variables.* These are the design variables the parameters of which are to be chosen by the optimization program to optimize the objective function (minimize the cost function, etc.). They define the dimensions of the work being designed, such as height, diameter, thickness, length, width, cross sections, etc.

$\boldsymbol{\eta}$: *Non-optimization design variables.* These are the set of variables the mean or characteristic values of which are fixed by the engineer or the code and must be given as data to the optimization program. Some examples are costs, material properties (unit weights, strength, Young modula, etc.), and other geometric dimensions of the work being designed.

$\boldsymbol{\kappa}$: *Random model parameters.* These are the set of parameters defining the random variability and dependence structure of the variables involved. For example, standard deviations, correlation coefficients, etc.

$\boldsymbol{\psi}$: *Auxiliary or non-basic variables.* These are auxiliary variables the values of which can be obtained from the basic variables $\boldsymbol{d}$ and $\boldsymbol{\eta}$, using certain formulas. They are used to facilitate the calculations and the statement of the problem constraints.

It should also be noted that variables in sets $\boldsymbol{d}$ and $\boldsymbol{\eta}$ are considered random, and, therefore, the corresponding means of $\boldsymbol{d}$ will be denoted $\bar{\boldsymbol{d}}$, and the mean or the characteristic values of $\boldsymbol{\eta}$ is denoted $\tilde{\boldsymbol{\eta}}$. In this paper deterministic variables are considered as particular cases of random variables.

Thus, the most general reliability-based design problem can be stated as follows:

$$\underset{\bar{\boldsymbol{d}}}{\text{minimize}}\ c(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) + c_{p_f}(\boldsymbol{\beta}), \tag{3}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}; F_i^0) \geq 0;\ \forall i \in I \tag{4}$$

$$\beta_i(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \geq \beta_i^0;\ \forall i \in I \tag{5}$$

$$\boldsymbol{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) = \boldsymbol{\psi} \tag{6}$$

$$r_j(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) \leq 0;\ \forall j \in J, \tag{7}$$

where the bars and tildes refer to mean and characteristic values of the variables, respectively, $c(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}})$ is the initial construction cost, $c_{p_f}(\boldsymbol{\beta})$ is the maintenance or repair function which depends on reliability indices $\boldsymbol{\beta}$ (related to the probabilities of failure), (4) are the limit state equations related to the different failure modes based on global safety-factors F_i^0, (5) are constraints that fix the lower bounds on the reliability indices β_i (this constraint could be expressed in terms of probabilities of failure), (6) are the equations that allow the auxiliary variables $\boldsymbol{\psi}$ to be obtained from the basic variables $\bar{\boldsymbol{d}}$ and $\tilde{\boldsymbol{\eta}}$, and the r_i in (7) are the geometric or code constraints. Note that in the most general case problem (3)-(7) uses mean or characteristic values of the random variables involved, and its solution provides mean or characteristic values of the design variables.

Rather than use approximate (and numerical) methods to perform the integration required to obtain the probabilities of failure, the FORM approach is used. This method has been widely utilized in engineering design [30], [31], [32], [33], [34], etc. It consists of transforming the integration problem into an optimization problem:

$$\beta_i(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \underset{\boldsymbol{d}_i, \boldsymbol{\eta}_i}{\text{Minimum}} \ \beta_i = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{8}$$

subject to

$$g_i(\boldsymbol{d}_i, \boldsymbol{\eta}_i, \boldsymbol{\psi}; 1) = 0 \tag{9}$$
$$T(\boldsymbol{d}_i, \boldsymbol{\eta}_i; \bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \mathbf{z} \tag{10}$$
$$h(\boldsymbol{d}_i, \boldsymbol{\eta}_i) = \boldsymbol{\psi}, \tag{11}$$

where $\boldsymbol{d}_i$ and $\boldsymbol{\eta}_i$ are the design points associated with the design $\boldsymbol{d}$ and $\boldsymbol{\eta}$ random variables for failure mode i, (9) is the limit state condition defining strict failure ($F_i^0 = 1$), and (10) is the usual transformation [35,36] that converts $\boldsymbol{d}_i$ and $\boldsymbol{\eta}_i$ into the standard independent normal random variable set $\mathbf{z}$ with components $(z_1, z_2, \ldots, z_n)$.

The probability of failure p_f is related to the reliability indices by the approximate relation $p_f = \Phi(-\beta)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable.

Thus, depending on the equations considered in problem (3)-(7), different decomposition techniques exist to solve the problem.

2.1 The failure-probability safety-factor method

To incorporate the advantages of both the optimal classic design and the optimal probability design, in this section the failure-probability safety-factor (FPSF) method is presented and solved by an iterative scheme (see [17]).

As indicated previously, the target of the FPSF method is:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \ c(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) \tag{12}$$

subject to constraints (4)-(7), where the cost term related to the probability of failure has been removed from the objective function. The method used to solve this problem, which is presented in this section, proceeds within an iterative scheme as follows:

Step 1. Solving the master problem.

An optimal classic design based on the safety-factor lower bounds using a first-order approximation of constraint (5) is done. In other words, problem (12), (4)-(7) is solved replacing (5) by the following constraint:

$$\beta_i^{(k)} + \boldsymbol{\lambda}_i^{(k)^T} (\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}^{(k)}) \geq \beta_i^0; \; i \in I, \tag{13}$$

where $\boldsymbol{\lambda}_i^{(k)^T}$ is the vector of the partial derivatives of $\beta_i^{(k)}$ with respect to $\bar{\boldsymbol{d}}^{(k)}$, which will be obtained below from the dual variables of the problem, and $k = \nu - 1$, where ν refers to the iteration number.

This problem is called the master problem. The result of this process is a set of values of the design variables for the actual iteration ($\bar{\boldsymbol{d}}^{(\nu)}$), which satisfy the safety-factor constraints (4), the geometric and code ones (7) and an approximation of the reliability constraints (5). Note that this method is analogous to the successive linear programming (SLP) approach for solving nonlinear programming problems [37], where only the reliability constraints are linearized at every iteration.

Step 2. Evaluating new β-values (subproblems).

The actual β-values associated with all modes of failure are evaluated, based on the values of the design variables obtained in Step 1. To this end, the problem (8)-(11) is solved for any $i \in I$ including the constraint:

$$\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}}^{(\nu)} : \boldsymbol{\lambda}^{(\nu)}, \tag{14}$$

where the colon after a constraint is used in this paper to indicate the dual variables associated with the constraint, that is, $\boldsymbol{\lambda}$ is the dual variable related to constraint (14), which provides the sensitivity (partial derivative) of the reliability index with respect to the actual values of the mean or characteristic design variable values ($\bar{\boldsymbol{d}}^{(\nu)}$). These derivatives are used to build constraint (13) for the next iteration.

In this step as many optimization problems as the number of modes of failure are solved, and the *design points* or *points of maximum likelihood* $\boldsymbol{d}_i^*$ and $\boldsymbol{\eta}_i^*$ for each mode of failure I are also obtained.

Step 3: Convergence checking.

If the relative change in the solution is smaller than a pre-specified threshold, i.e., if the norm $||\bar{\boldsymbol{d}}^{(\nu)} - \bar{\boldsymbol{d}}^{(\nu-1)}|| \leq \varepsilon$, the process stops and the optimal solution has been found, otherwise the procedure continues in **Step 1**.

The process of solving iteratively these two problems is repeated starting from $\nu = 0$ and increasing the value of ν in one unit, until convergence of the solution is obtained. Note that at iteration $\nu = 0$ there is no hyperplane approximation (13) of constraint (5). This solution is a lower bound of the global problem because it is less constrained.

It should be noted that replacing (5) by (13) relaxes problem (12), (4)-(7) in the sense that functions $\beta_i(\cdot)$ are approximated using cutting hyperplanes.

Once the convergence of the process has been attained, one can calculate:

(1) The actual safety-factors F_i because the values F_i^0 are only lower bounds, but not actual values.
(2) The actual failure mode probabilities.

Finally, a Monte Carlo simulation using importance sampling can be performed to check the goodness of the approximations and the correlation between different failure modes, i.e., system probabilities of failure.

In Appendix A, the convergence problem of the iterative method is discussed.

2.2 Cost function depending on the probability of failure

Alternatively, the design can be based on minimizing the expectation of the total cost. The objective function consists of two components that describe the construction costs as a function of design variables, and the expected costs of failure [21]:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}}\ c(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) + c_{p_f}(\boldsymbol{\beta}) \tag{15}$$

subject to (6)-(7), where constraints related to the safety-factor and reliability index lower bounds are removed. The optimal solution of this problem also provides the optimal values of the target reliability indexes.

Decomposition techniques [38], which have been already applied to reliability optimization problems by [39], are ideal for solving this type of problem. However, one has to pay the price of iteration. With this technique, the complex

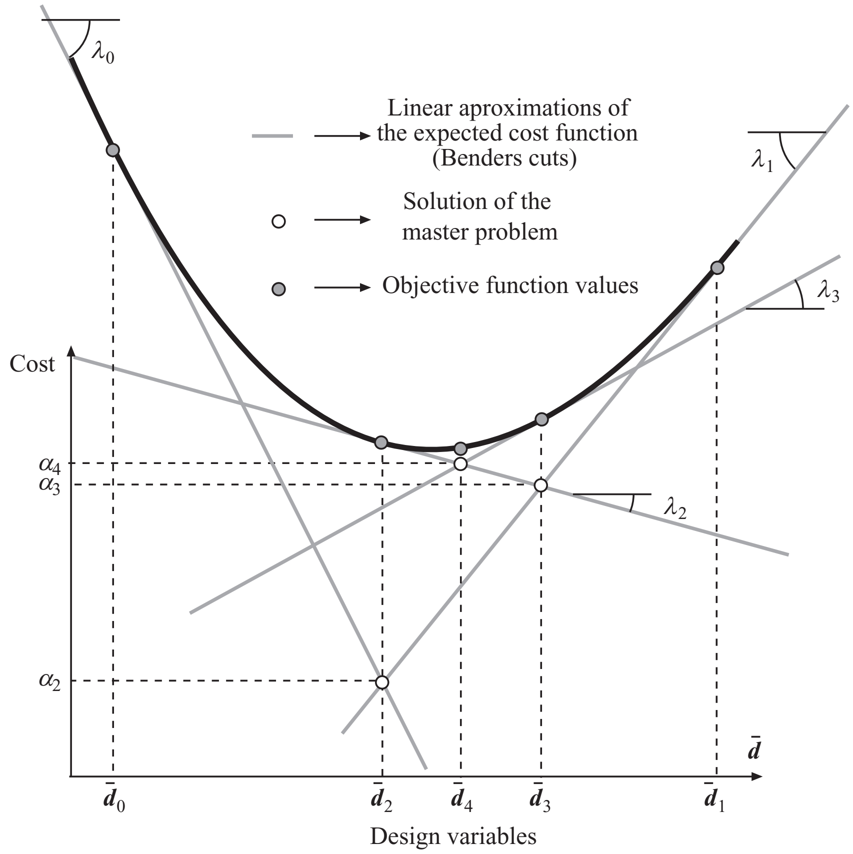

Fig. 1. Graphical illustration of how the expected cost function is approximated using Benders cuts.

original problem is replaced by two simpler problems, which are solved iteratively. The first is the master problem (approximation of the original one) and the second is the subproblem or subproblems (evaluation of the probabilities of failure using FORM), so that the results of the subproblems are being progressively taken into account in the master problem. The main idea consists of approximating the expected cost function by an increasing number of hyperplanes (see Figure 1).

The following iterative scheme based on Benders decomposition can be applied to solve the problem (15), (6)-(7):

- **Step 0: Initialization.** First, the iteration counter $\nu = 1$ is initialized, the auxiliar variable α is set to its initial lower bound α^{lo} and some initial values for the mean or characteristic values of the design variables $\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}}^{(1)}$ are selected.

- **Step 1: Subproblem solution.** The subproblems, i.e., the problem (8)-(11) is solved for any $i \in I$ including the constraint:

$$\bar{\boldsymbol{d}} = \bar{\boldsymbol{d}}^{(\nu)} : \boldsymbol{\mu}^{(\nu)}, \tag{16}$$

i.e., in this step as many optimization problems as the number of modes of failure are solved, where $\boldsymbol{\mu}^{(\nu)} = \partial \boldsymbol{\beta}(\bar{\boldsymbol{d}}^{(\nu)}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa})/\partial \bar{\boldsymbol{d}}^{(\nu)}$ are the dual variables associated with constraint (16).

Next, evaluate the expected cost function in (15) which, hereinafter is denoted as $\alpha(\bar{\boldsymbol{d}}^{(\nu)})$:

$$\alpha(\bar{\boldsymbol{d}}^{(\nu)})) = c(\bar{\boldsymbol{d}}^{(\nu)}, \tilde{\boldsymbol{\eta}}) + c_{p_f}(\boldsymbol{\beta}^{(\nu)}), \tag{17}$$

representing the expected total cost for fixed values of the optimizing design variables (see the corresponding gray shadow point of the expected total cost function in Figure 1).

Next, the objective function (total expected cost) in (15) is approximated using the derivatives of the objective cost function, which are obtained using the chain rule as follows:

$$\boldsymbol{\lambda}^{(\nu)} = \frac{\partial c(\bar{\boldsymbol{d}}^{(\nu)}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})}{\partial \bar{\boldsymbol{d}}^{(\nu)}} + \frac{\partial c_{p_f}(\boldsymbol{\beta}^{(\nu)})}{\partial \boldsymbol{\beta}^{(\nu)}} \boldsymbol{\mu}^{(\nu)^T}. \tag{18}$$

Note that these values correspond to the slopes of the approximating hyperplanes in Figure 1.

- **Step 2: Convergence checking.** First, an upper bound of the objective function optimal value is computed as $z_{\text{up}}^{(\nu)} = \alpha(\bar{\boldsymbol{d}}^{(\nu)})$, taking into account that function (17) is more constrained than the objective function of the original problem (15), (6)-(7) in the sense that the optimization variables are fixed for the actual iteration. Similarly, a lower bound of the objective function optimal value is obtained by $z_{\text{down}}^{(\nu)} = \alpha$. Then, if $\left| \frac{z_{\text{up}}^{(\nu)} - z_{\text{down}}^{(\nu)}}{z_{\text{up}}^{(\nu)}} \right|$ is lower than the tolerance ε, the procedure stops, otherwise, it goes to Step 3. An alternative stopping criteria could be used, that is, if the norm $||\bar{\boldsymbol{d}}^{(\nu)} - \bar{\boldsymbol{d}}^{(\nu-1)}|| \leq \varepsilon$.

- **Step 3: Master problem solution for iteration ν.** An approximation of the original problem, the so-called master problem, where the objective function is replaced by approximating hyperplanes is solved:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \ \alpha \tag{19}$$

subject to

$$\alpha \geq \alpha(\bar{\boldsymbol{d}}^{(k)}) + \boldsymbol{\lambda}^{(k)^T}(\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}^{(k)}); k \in K \tag{20}$$

$$\boldsymbol{\psi} = \boldsymbol{q}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) \tag{21}$$

$$\mathbf{0} \leq \mathbf{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) \tag{22}$$

$$\alpha \geq \alpha^{\text{lo}}, \tag{23}$$

where $K = 1, \cdots, \nu - 1$, obtaining the new values of the mean or characteristic design variable values $\bar{\boldsymbol{d}}^{(\nu)}$ (see the white shadow intersecting points between hyperplanes in Figure 1). Note that the objective function of the original problem in (15) is replaced by the linear approximations in (20) (see the Benders cuts in Figure 1). As problem (19)-(23) is a relaxation of the original problem (15), (6)-(7) the optimal solution of this problem (α) is a lower bound of the objective function optimal value.

Let $\nu = \nu + 1$, and go to Step 1 and the process is repeated until convergence.

The proposed algorithm provides the solution of the problem in a finite number of iterations whenever the objective function is convex, otherwise, the procedure fails to converge [40]. In Appendix B, the convergence of this algorithm is proved.

2.3 Mixed approach

The last method consists of solving the most general reliability based problem (3)-(7). In this case the algorithm is a combination of both algorithms previously stated in Subsections 2.1 and 2.2, where the Benders decomposition algorithm is also used, replacing the master problem (19)-(23) by:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}}\ \alpha \tag{24}$$

subject to

$$
\begin{aligned}
\alpha \geq & \alpha(\bar{\boldsymbol{d}}^{(k)}) + \boldsymbol{\lambda}^{(k)^T}(\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}^{(k)}); k \in K && (25)\\
g_i(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}; F_i^0) \geq & 0;\ \forall i \in I && (26)\\
\beta_i^{(k)} + \boldsymbol{\lambda}_i^{(k)^T}(\bar{\boldsymbol{d}} - \bar{\boldsymbol{d}}^{(k)}) \geq & \beta_i^0;\ i \in I;\ k = \nu - 1, && (27)\\
\boldsymbol{\psi} = & \boldsymbol{q}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}) && (28)\\
\mathbf{0} \leq & \mathbf{h}(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi}) && (29)\\
\alpha \geq & \alpha^{\text{lo}}, && (30)
\end{aligned}
$$

where two additional constraints (26) and (27) have been included, which correspond to the reliability constraints (5) linearized at every iteration and to the constraint fixing the lower bounds of the global safety-factors, respectively.

3 SENSITIVITY ANALYSIS

Sensitivity analysis is the study of how the variation in the output of a model can be apportioned, qualitatively or quantitatively, to different sources of variation, and attempts to determine how the model depends upon the data or information fed into it, upon its structure and upon the framing assumptions made to build it. As a whole, sensitivity analysis is used to increase the confidence in the model and its predictions, by providing an understanding of how the model response variables respond to changes in the inputs. Adding a sensitivity analysis to a study means adding quality to it.

Even though a sensitivity analysis is not a standard procedure, it is very useful to (a) the designer, who can discover which data values have greater influence on the safety and cost of the designed work, (b) to the builder, who can learn how changes in prices or dimensions influence the total safety and cost, and (c) to the code maker, who can find out the costs and reliability changes associated with an increase or decrease in the required safety-factors or failure probabilities they select. Below, a methodology is proposed and which is simple, efficient and allows a simultaneous determination of all the sensitivities. At the same time it is the natural way of evaluating sensitivities when design is based on optimization procedures.

The sensitivity analysis problem in reliability based optimization has been discussed by several authors, such as [41] or [42]. It is shown in this section how duality methods can be applied to sensitivity analysis in a straightforward and simple manner (see Castillo et al. [22,43,44]).

The proposed method is based on including additional variables and constraints fixing their value to the values of the parameters whose sensitivities are sought, i. e.:

$$\boldsymbol{\eta} = \tilde{\boldsymbol{\eta}} : \boldsymbol{\omega}. \tag{31}$$

Because constraints (31) involve the data in their right hand sides, and the dual variables $\boldsymbol{\omega}$ are the sensitivities of the objective function value to changes in the constraints right hand side terms, the desired sensitivities can be obtained by printing the values of the corresponding dual variables. In other words, the values of the dual variables associated with the constraints in (31), give how much the objective function changes with a very small unit increase in the corresponding data parameter.

This method can be applied to every optimization problem, either master or subproblem, but for the total expected cost problem. In this case sensitivities must be obtained using the chain rule analogously as in (18):

$$\frac{\partial C_{\text{to}}}{\partial \boldsymbol{\eta}} = \frac{\partial c(\bar{\boldsymbol{d}}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\psi})}{\partial \boldsymbol{\eta}} + \frac{\partial c_{p_f}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \boldsymbol{\omega}^T. \tag{32}$$

where $\boldsymbol{\omega}$ is the dual variable associated with constraint (31) included in the subproblem (8)-(11).

4 APPLICATIONS

These methods have been successfully applied to different civil engineering works, such as:

(1) A bridge crane design, retaining walls designed to hold back soil where abrupt changes in ground elevation occur, a composite beam design, and design of rubble-mound breakwaters, all of them minimizing the construction cost using the failure-probability safety-factor method [19,18,45].
(2) A composite breakwater design based on minimizing initial/construction costs subject to yearly failure rate bounds for all failure modes [20].
(3) A composite breakwater design based on minimizing the total expected cost during the lifetime of the structure [21]
(4) An application of decomposition techniques for solving the inverse reliability problem [39].

In what follows a brief description of how the proposed methods for the reliability-based optimization of a rubblemound breakwater considering overtopping failure can be used, is described.

Consider the construction of a rubblemound breakwater (see Figure 2) to protect a harbor area from high waves during storms. The crest must be high enough to prevent the intrusion of sea water into the harbor by overtopping. For simplicity, only overtopping failure is considered.

The goal is to compare an optimal design of the breakwater based on the different proposed methods.

The construction C_{co} and insurance cost C_{in} functions are: $C_{\text{co}} = c_{\text{c}}v_{\text{c}} + c_{\text{a}}v_{\text{a}}$, and $C_{\text{in}} = 5000 + 1.25 \times 10^6 P_{\text{f}}^{D^2}$, respectively, where v_{c} and v_{a} are the concrete and armor volumes, respectively, c_{c} and c_{a} are the respective construction costs per unit volume, and P_{f}^{D} is the probability of overtopping failure during the design sea state for a given breakwater lifetime, D.

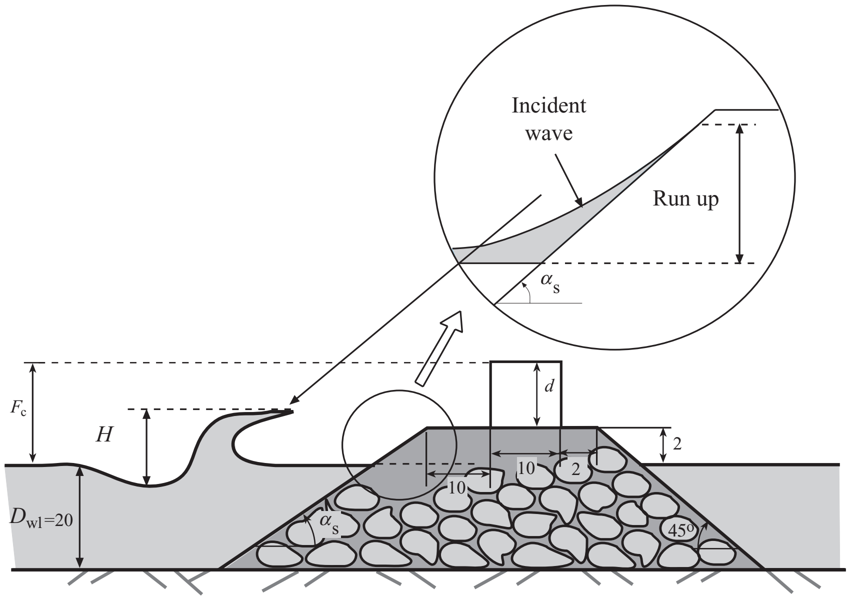

Fig. 2. Parameterized rubblemound breakwater used in the example.

With this approximation, overtopping (failure) occurs whenever the difference between the maximum excursion of water over the slope, R_u, called wave run-up, exceeds the freeboard F_c, i. e., if $F_c - R_u < 0$.

The set of variables and parameters involved in this problem can be partitioned into the subsets shown in Section 2: optimization design variables $\boldsymbol{d} = \{F_c, \tan\alpha_s\}$ whose values must be selected by the optimization procedure that in this case are considered deterministic. Non-optimization design variables $\boldsymbol{\eta} = \{A_u, B_u, D_{wl}, g, H, T, c_c, c_a\}$, which are used as data. In this set the only random variables considered are the wave height H and period T. A_u and B_u are given coefficients depending on the armor units to calculate run-up, D_{wl} is the design water level, and g is the gravity constant. The random model parameters $\boldsymbol{\kappa} = \{H_s, \bar{T}, d_{st}\}$ define the random variability of the wave height and period within the design sea state, where H_s is the significant wave height, $\bar{T}$ is the mean period, and d_{st} is the duration of a sea state. The non-basic variables set is as follows $\boldsymbol{\psi} = \{I_r, v_a, v_c, C_{co}, C_{in}, R_u, L, d\}$, where I_r is the Iribarren number, L is the wave length and d is the caisson height.

For a rubblemound breakwater of slope $\tan\alpha_s$ and freeboard F_c (see Figure 2), the most general reliability based design problem (3)-(7) consists of:

$$\underset{F_c, \tan\alpha_s}{\text{Minimize}} \; C_{to} = c_c v_c + c_a v_a + 5000 + 1.25 \times 10^6 {P_f^D}^2$$

subject to

$$F_c/R_u \geq F^0 \tag{33}$$

$$P_f^D \leq P_f^0, \tag{34}$$

$$F_{\mathrm{c}} = 2 + d \tag{35}$$
$$v_{\mathrm{c}} = 10d \tag{36}$$
$$v_{\mathrm{a}} = \frac{1}{2}(D_{\mathrm{wl}} + 2)\left(46 + D_{\mathrm{wl}} + \frac{D_{\mathrm{wl}} + 2}{\tan\alpha_{\mathrm{s}}}\right) \tag{37}$$
$$\frac{R_{\mathrm{u}}}{\tilde{H}} = A_{\mathrm{u}}\left(1 - e^{B_{\mathrm{u}} I_{\mathrm{r}}}\right) \tag{38}$$
$$I_{\mathrm{r}} = \frac{\tan\alpha_{\mathrm{s}}}{\sqrt{\tilde{H}/L}} \tag{39}$$
$$\left(\frac{2\pi}{\tilde{T}}\right)^2 = g\frac{2\pi}{L}\tanh\frac{2\pi D_{\mathrm{wl}}}{L} \tag{40}$$
$$P_{\mathrm{f}} = \Phi(-\beta) \tag{41}$$
$$P_{\mathrm{f}}^{\mathrm{D}} = 1 - (1 - P_{\mathrm{f}})^{(d_{\mathrm{st}}/\bar{T})} \tag{42}$$
$$1/5 \le \tan\alpha_{\mathrm{s}} \le 1/2 \tag{43}$$

where constraints (33) and (34) correspond to (4) and (5), respectively. Note that (34) is written in terms of failure probabilities. Constraints (35)-(42) correspond to (6), and (43) corresponds to the geometric constraint (7). The random variable values for this particular problem are equal to the characteristic values $\tilde{H} = 1.8H_{\mathrm{s}}$ and $\tilde{T} = 1.1\bar{T}$. (38) is the equation, based on experiments, which allows the run-up R_{u} to be evaluated, and I_{r} is the Iribarren number given by (39) . Note that L is the wave length obtained from the dispersion equation (40). The objective of this problem is to minimize the total expected cost during lifetime fulfilling the reliability constraints given in terms of global safety-factors (33) and reliability indexes (probabilities of failure) (34).

Alternatively, the designer must be interested in minimizing the construction cost fulfilling the reliability constraints given in terms of global safety-factors (33) and reliability indexes (probabilities of failure) (34), i.e., problem (12), (4)-(7). In this case the objective function of the previous problem is replaced by

$$\underset{F_{\mathrm{c}},\, \tan\alpha_{\mathrm{s}}}{\text{Minimize}}\ C_{\mathrm{co}} = c_{\mathrm{c}} v_{\mathrm{c}} + c_{\mathrm{a}} v_{\mathrm{a}}\,.$$

The last option considered corresponds to the minimization of the total expected cost during lifetime (15), (6)-(7). Note that the global safety-factor and reliability constraints (33) and (34) and equations (38)-(40) needed to calculate the run-up for given characteristic values of the random variables are removed from this model. The objective of this problem is to get the values of the design variables that minimize the total expected cost, and, therefore, the optimal probability of failure is obtained as well.

In both models P_{f} is the probability of overtopping failure due to a single wave, $N = d_{\mathrm{st}}/\bar{T}$ is the mean number of waves during the design sea state for lifetime D, and d_{st} is its duration. These probabilities are obtained through the reliability index β solving the following optimization problem:

$$\underset{H,T}{\text{minimize}}\ \beta = \sqrt{z_1^2 + z_2^2},$$

subject to

$$\frac{R_\mathrm{u}}{H} = A_\mathrm{u}\left(1 - e^{B_\mathrm{u} I_\mathrm{r}}\right) \tag{44}$$

$$I_\mathrm{r} = \frac{\tan\alpha_\mathrm{s}}{\sqrt{H/L}} \tag{45}$$

$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{L}\tanh\frac{2\pi D_\mathrm{wl}}{L} \tag{46}$$

$$\Phi(z_1) = 1 - e^{-2(H/H_\mathrm{s})^2} \tag{47}$$

$$\Phi(z_2) = 1 - e^{-0.675(T/\bar{T})^4} \tag{48}$$

$$F_\mathrm{c} = R_\mathrm{u}, \tag{49}$$

where (44)-(46) correspond to (11). The basic random variables in this problem are H and T, which are assumed to be independent; note that actual values are used instead of characteristic values. The corresponding Rosenblatt transformation (10) is given by (47)-(48) where z_1 and z_2 are independent standard normal random variables, and (49) is the limit state equation forcing strict failure.

Assuming the following values for the variables and parameters involved:

$$\begin{array}{llll} D_\mathrm{wl} = 20\ m; & A_u = 1.05; & B_u = -0.67; & c_\mathrm{c} = 60\ \$/m^3; \\ c_\mathrm{a} = 2.4\ \$/m^3; & g = 9.81\ m/s^2; & H_\mathrm{s} = 5\ m; & \bar{T} = 10\ s; \\ d_\mathrm{st} = 1\ h; & P_\mathrm{f}^0 = 0.001; & F^0 = 1.2; & \varepsilon = 10^{-3}, \end{array}$$

where ε is the tolerance, and using the methods proposed in Sections 2.1, 2.2 and 2.3, the above problems are solved.

The optimization problem has been solved using solver CONOPT [46] under the General Algebraic Modeling System (GAMS) [47], which is a high-level modeling system for mathematical programming and optimization. It consists of a language compiler and a stable of integrated high-performance solvers. GAMS is tailored for complex, large scale modeling applications, and makes it possible to build large maintainable models that can be adapted quickly to new situations.

The solution for all models is provided in Table 1. The failure-probability safety-factor method converges in 8 iterations within the admissible tolerance.

Table 1
Solution of the three different models for the illustrative example.

Iterations	F_c	$\tan\alpha_s$	β	P_{f}	$P_{\mathrm{f}}^{\mathrm{D}}$	F	error
Mixed approach							
16	5.971	0.239	4.680	1.46×10^{-6}	5.26×10^{-4}	1.303	8×10^{-4}
Failure-probability safety-factor method							
8	5.903	0.240	4.546	2.78×10^{-6}	1×10^{-3}	1.342	4×10^{-4}
Expected cost							
18	5.959	0.239	4.681	1.46×10^{-6}	5.26×10^{-4}	–	1×10^{-3}

Note that the probability of failure during lifetime coincides with the maximum required tolerance $P_{\mathrm{f}}^{0} = 0.001$, which means that constraint (34) is active, whereas the value of the global safety-factor is $F = 1.342$, i.e., constraint (33) is inactive. This means that the reliability-based constraint is more restrictive than the classical safety-factor equation. Construction cost is $C_{\mathrm{co}}^{*} = \$6508.29$.

The solution that minimizes the total expected cost is $C_{\mathrm{co}}^{*} = \$6552.85$, $C_{\mathrm{in}}^{*} = \$5034.55$, and $C_{\mathrm{to}}^{*} = \$11587.41$. Variable values are also provided in Table 1, where the following optimal reliability values are obtained $\beta^{*} = 4.681$; $P_{\mathrm{f}}^{*} = 1.46\times10^{-6}$ and $P_{\mathrm{f}}^{\mathrm{D}*} = 5.26\times10^{-4}$. Note that this design provides higher construction costs, which implies lower probabilities of failure. In Figure 3 the evolution of the lower and upper bound during the iterative process up to 15 iterations is shown. Note that bounds converge to the optimal solution in 18 iterations within the admissible tolerance.

The mixed approach provides almost the same values as the minimization of the total expected cost because the additional constraints related to the lower bounds on global safety-factors and reliability indexes are not active. To get the same result, tolerance must be decreased.

With respect to the sensitivity analysis, the sensitivities of the reliability index due to overtopping with respect the freeboard and the slope angle are:

Failure-probability safety-factor method:

$$\frac{\partial\beta}{\partial F_c} = 1.70;\ \frac{\partial\beta}{\partial\tan\alpha_s} = -28.76;\ F_c\frac{\partial\beta}{\partial F_c} = 10.07;\ \tan\alpha_s\frac{\partial\beta}{\partial\tan\alpha_s} = -6.89, \tag{50}$$

Total expected cost:

$$\frac{\partial\beta}{\partial F_c} = 1.93;\ \frac{\partial\beta}{\partial\tan\alpha_s} = -32.84;\ F_c\frac{\partial\beta}{\partial F_c} = 11.51;\ \tan\alpha_s\frac{\partial\beta}{\partial\tan\alpha_s} = -7.83,$$

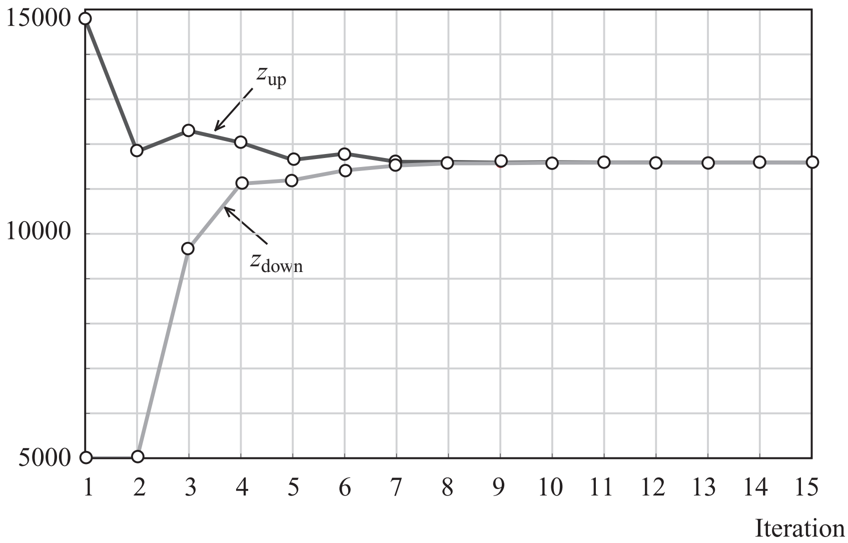

Fig. 3. Evolution of the objective function upper and lower bounds.

where the last two ones in every row are relative sensitivities. Note that on increasing the freeboard F_c, the reliability index increases and the probability of failure decreases, whereas on increasing the slope $\tan \alpha_s$ the reliability index decreases and the probability of overtopping increases. Note also that the probability of overtopping is more sensitive to the freeboard because the relative sensitivity is higher.

The reliability of the solution given by the failure-probability safety-factor method is more sensitive with respect to the slope angle and less sensitive with respect to the freeboard than the total expected minimization solution because the corresponding relative sensitivities are higher and lower, respectively.

With respect to cost sensitivities, the sensitivity of the construction cost with respect to the reliability index lower bound for failure-probability safety-factor method design is $\frac{\partial C_{\text{co}}}{\partial \beta_0} = 351.838$, which means that one unit increase in the reliability index lower bound β_0 (lower probability of failure) increases the construction cost by \$351.838. The derivative of the total expected cost with respect to the design water level using (32) is $\frac{\partial C_{\text{to}}}{\partial D_{\text{wl}}} = 356.02$, i.e., if the design water level is increased by 1 meter, the total expected cost increases by \$356.02.

5 CONCLUSIONS

In this paper a useful methodology has been presented which provides a rational and systematic procedure for automatic and optimal design of several reliability-based optimization problems using decomposition techniques. The method is ideal for a sensitivity analysis, which can be easily performed by transforming the input parameters into auxiliary variables, which are set to their associated actual values.

Some additional advantages of the proposed method are:

(1) The method can be easily implemented using standard optimization frameworks, such as GAMS, for example.
(2) It can be applied to different types of problems such as linear, non-linear, mixed-integer problems. The designer merely needs to choose the adequate optimization algorithm.
(3) The failure-probability safety-factor method for engineering design provides a double way of safety control, safety-factors and failure probabilities, and interesting calibration possibilities for the classic and probability-based designs.
(4) Since safety-factors and probabilities of failure are dealt with, the method enables there to be communication between classical and probability-based designers.
(5) The proposed method takes full advantage of the optimization packages, in the sense that:
    (a) It makes the solution of huge problems possible without the need to be an expert in optimization techniques.
    (b) The constraints need not be written in terms of the design variables. Auxiliary or intermediate variables can be used.
    (c) The cost function and the constraints need not be written in explicit form, i.e. auxiliary variables and equations can be used to facilitate the statement of the problem.
    (d) The failure region need not be written in terms of the normalized (transformed) variables. The transformation equation, in direct or inverse form, is sufficient.
    (e) The responsibility for iterative methods is given to the optimization software.
(6) Sensitivity values are given, for free, if one converts the data values into artificial variables, by printing the values of the dual problem.
(7) It leads to an *automatic, optimal and designer independent design*, i.e., the values of the design variables are given, not by the engineer, but by the optimization process itself. Note that this statement does not mean that engineering judgement is not needed anymore; on the contrary, it is extremely important in order to establish the appropriate constraints on the models.
(8) It makes a specific code calibration for each work being designed possible.

The proposed methods can be improved by using importance sampling and SORMS (see Rackwitz [48], derKiureghian at al. [49]), which is the aim of future research by the authors. Additionally, further research must be done regarding system reliability evaluations. Approximate methods, using the information contained in the solutions of the decoupled problems, such us those proposed by [50] and [51], seem to be plausible alternatives for the coming future.

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A Approximating Hyperplane Convergence Analysis

Problem (12), (4)-(7) can be reformulated in a simpler manner as follows:

$$\underset{\boldsymbol{x}}{\text{minimize}}\ f(\boldsymbol{x}) \tag{A.1}$$

subject to

$$h_i(\boldsymbol{x}) = 0;\ i = 1, \ldots, \ell \tag{A.2}$$
$$g_i(\boldsymbol{x}) \leq 0;\ i = 1, \ldots, m, \tag{A.3}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is the vector of the decision variables, $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function, and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^\ell$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$ are equality and inequality constraints, respectively. Note that $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x}))^T$ are the inequality constraints representing the reliability index bounds the evaluation of which requires the solution of an optimization problem per constraint (subproblems).

Considering that problem (A.1)-(A.3) is feasible and that the functions involved are doubly continuously differentiable. The first-order KKT optimality conditions for problem (A.1)-(A.3) require a primal solution $\boldsymbol{x}$ and two Lagrange multiplier vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ associated with the equality and inequality constraints, respectively, such that:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \nabla f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \nabla \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{\mu}^T \nabla \boldsymbol{g}(\boldsymbol{x}) = 0 \tag{A.4}$$
$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.5}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0} \tag{A.6}$$
$$\boldsymbol{g}(\boldsymbol{x})\boldsymbol{\mu}^T = \mathbf{0} \tag{A.7}$$
$$\boldsymbol{\mu} \geq \mathbf{0}. \tag{A.8}$$

The Newton-Raphson method can be used to solve this system of equations, establishing the first-order condition that equates the gradient to zero. Starting from initial values $\boldsymbol{x}_k$, $\boldsymbol{\lambda}_k$ and $\boldsymbol{\mu}_k$, the search directions are obtained iteratively solving the following system of equations:

$$\nabla^2\mathcal{L}_k(\boldsymbol{x}-\boldsymbol{x}_k) + \boldsymbol{\lambda}^T\nabla\boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{\mu}^T\nabla\boldsymbol{g}(\boldsymbol{x}_k) = -\nabla f(\boldsymbol{x}_k) \tag{A.9}$$
$$\nabla\boldsymbol{h}(\boldsymbol{x}_k)(\boldsymbol{x}-\boldsymbol{x}_k) = -\boldsymbol{h}(\boldsymbol{x}_k) \tag{A.10}$$
$$\mu_i[g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^T(\boldsymbol{x}-\boldsymbol{x}_k)] = 0;\ \forall i \tag{A.11}$$
$$\boldsymbol{\mu} \geq \mathbf{0}. \tag{A.12}$$

It can be shown [37] that if $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ is a regular KKT solution to the original problem (A.1)-(A.3) satisfying the second-order sufficiency conditions and if $\boldsymbol{x}_k$, $\boldsymbol{\lambda}_k$ and $\boldsymbol{\mu}_k$ is initialized sufficiently close to the optimal solution, then the foregoing iterative process will converge to the optimal solution.

If all the functions involved in the original problem (A.1)-(A.3) and their gradients could be evaluated explicitly at every Newton step, the optimal solution would be attained by iteratively solving system (A.9)-(A.12). However, constraints $\boldsymbol{g}(\boldsymbol{x}_k)$ and $\nabla\boldsymbol{g}(\boldsymbol{x}_k)$ at a given point $\boldsymbol{x}_k$ are calculated by solving the subproblems (FORM), which are optimization problems themselves. Then, the convergence values $\boldsymbol{x}_{k+1}$ satisfying (A.9)-(A.12) for fixed values of $\boldsymbol{g}(\boldsymbol{x}_k)$ and $\nabla\boldsymbol{g}(\boldsymbol{x}_k)$ are obtained solving the original problem but replacing (A.3) by a linear approximation $\boldsymbol{g}(\boldsymbol{x}_k) + \nabla\boldsymbol{g}(\boldsymbol{x}_k)^T(\boldsymbol{x}-\boldsymbol{x}_k)$, in this case the convergence is achieved using an optimization algorithm. Equivalently, system (A.9)-(A.12) could be solved iteratively for fixed values of $\boldsymbol{g}(\boldsymbol{x}_k)$ and $\nabla\boldsymbol{g}(\boldsymbol{x}_k)$ without re-evaluating this values at every Newton step.

For the new values of the variables $\boldsymbol{x}_{k+1}$, which are the optimal solution of problem (A.1)-(A.3) but replacing (A.3) by a linear approximation, and solving the subproblems calculates a new linear approximation, i.e. $\boldsymbol{g}(\boldsymbol{x}_{k+1}) + \nabla\boldsymbol{g}(\boldsymbol{x}_{k+1})^T(\boldsymbol{x}-\boldsymbol{x}_{k+1})$. Thus, the original problem using the new linear approximation of the inequality constraints is solved, which is equivalent to getting values such that they satisfy the system of equations:

$$\nabla^2\mathcal{L}_{k+1}(\boldsymbol{x}-\boldsymbol{x}_{k+1}) + \boldsymbol{\lambda}^T\nabla\boldsymbol{h}(\boldsymbol{x}_{k+1}) + \boldsymbol{\mu}^T\nabla\boldsymbol{g}(\boldsymbol{x}_{k+1}) = -\nabla f(\boldsymbol{x}_{k+1}) \tag{A.13}$$
$$\nabla\boldsymbol{h}(\boldsymbol{x}_{k+1})(\boldsymbol{x}-\boldsymbol{x}_{k+1}) = \mathbf{0} \tag{A.14}$$
$$\mu_i[g_i(\boldsymbol{x}_{k+1}) + \nabla g_i(\boldsymbol{x}_{k+1})^T(\boldsymbol{x}-\boldsymbol{x}_{k+1})] = 0;\ \forall i \tag{A.15}$$
$$\boldsymbol{\mu} \geq \mathbf{0}. \tag{A.16}$$

Solving the sequence of system of equations (A.9)-(A.12), (A.13)-(A.16) and so on, is equivalent to solving the system of equations (A.9)-(A.12) where at every Newton step the inequality constraints and their gradients are calculated by solving the subproblems. If problem (A.1)-(A.3) is feasible and has a KKT solution the approximate hyperplane algorithm converges to the same solution.

Note that extremely demanding reliability bounds, which are not physically possible to fulfil, can potentially make problem (A.9)-(A.12) infeasible.

B Benders Decomposition Convergence Analysis

Problem (15), (6)-(7) can be reformulated in a simpler manner as follows:

$$\underset{\boldsymbol{x}}{\text{minimize}}\ f(\boldsymbol{x}) \tag{B.1}$$

subject to

$$h_i(\boldsymbol{x}) = 0;\ i = 1, \ldots, \ell \tag{B.2}$$
$$g_i(\boldsymbol{x}) \le 0;\ i = 1, \ldots, m, \tag{B.3}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is the vector of the decision variables, $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function, and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^\ell$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$.

In this case the objective function (B.1) cannot be straightforwardly evaluated as it involves solving inner optimization problems. The master problem at iteration k is:

$$\underset{\alpha, \boldsymbol{x}}{\text{Minimize}}\ \alpha \tag{B.4}$$

subject to

$$\alpha \ge f(\boldsymbol{x}_k) + \boldsymbol{\lambda}_k^T(\boldsymbol{x} - \boldsymbol{x}_k); k \in K \tag{B.5}$$
$$h_i(\boldsymbol{x}) = 0;\ i = 1, \ldots, \ell \tag{B.6}$$
$$g_i(\boldsymbol{x}) \le 0;\ i = 1, \ldots, m, \tag{B.7}$$

where $\boldsymbol{\lambda}_k = \nabla f(\boldsymbol{x}_k)$.

By definition, $f(\boldsymbol{x})$ is convex if and only if $f(\boldsymbol{y}) \ge f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x})$ holds for all $\boldsymbol{x}$, $\boldsymbol{y} \in \text{domain} f(\boldsymbol{x})$. This condition is equivalent to constraint (B.5), then if $k \to \infty$ this constraint reproduces exactly the original objective function in the optimal solution neighborhood, which means that both problems are equivalent if and only if function $f(\boldsymbol{x})$ is convex in the feasibility domain defined by (B.2)-(B.3). In that case the original problem (B.1)-(B.3) and the master problem (B.4)-(B.7) are equivalent and converge to the same solution.

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