Locational Marginal Price Sensitivities

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***Abstract*— Within an optimal power flow market clearing framework, this paper provides expressions to compute the sensitivities of locational marginal prices with respect to power demands. Sensitivities with respect to other parameters can also be obtained. An example and a case study are used to illustrate the expressions derived.**

***Index Terms*— Locational marginal prices, Optimal power flow, Sensitivities.**

NOTATION

The main notation used throughout the paper is stated below for quick reference. Other symbols are defined as required in the text.

A. Functions

$f(\cdot)$ is the optimal power flow objective function.
$h_{Pi}(\cdot)$ is the active power injection at bus i.
$h_{Qi}(\cdot)$ is the reactive power injection at bus i.
$g_{Sij}(\cdot)$ is the complex power flow magnitude through line ij.

B. Variables

p_{Gi} is the active power generation at bus i.
q_{Gi} is the reactive power generation at bus i.
v_i is the voltage magnitude at bus i.
δ_i is voltage angle at bus i.

C. Multipliers

λ_i is the locational marginal price at bus i.

D. Constants

c_i is a network or generator related constant.
p_{Di} is the active power demand at bus i.
q_{Di} is the reactive power demand at bus i.
π is the number π.

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E. Sets

J is the set of active inequality constraints.
Ω_i is the set of buses adjacent to bus i.

F. Numbers

l is the number of equality constraints.
m is the number of inequality constraints.
m_J is the cardinality of J, i.e., the number of active inequality constraints.
n_B is the number of buses.

G. Indices

i, j are indices for buses.

It should be noted that a variable, function or parameter written in bold without index is a vector form representing the corresponding quantities. For example, the symbol $\boldsymbol{\delta}$ represents the vector of bus voltage angles. Moreover, a superscript "max" indicates upper bound, while a superscript "min" indicates lower bound.

I. INTRODUCTION

Electricity markets are reaching a reasonable level of maturity, as they have evolved during the last decade. Fully fledged markets, such as the New England ISO [1], the New York ISO [2] or the PJM Interconnection [3], use or are planning to use an Optimal Power Flow (OPF) as the market clearing tool.

Locational Marginal Prices (LMP) [4] are easily obtained within an OPF framework, as they are the sensitivities (dual variables) associated with the active power balance equations [5], [6]. Moreover, LMPs are generally recognized to be the most appropriate prices for electricity [7]. This is so for day-ahead markets as well as for shorter-term markets such as balancing markets. LMPs are used in the New England ISO [1], the New York ISO [2], the PJM Interconnection [3] and market operators such as Ontario IMO [8]. References [9] and [10] provide a in-depth analysis of LMPs.

If LMPs govern the electricity business, a fundamental question arises: How LMPs change as parameters change? Answering rigorously this question is the subject and novel contribution of this paper.

We analyze changes in LMPs with respect to operational parameters, i.e., demands, generator cost parameters and voltage bounds. Note, however, that changes with respect to line design parameters (resistance, reactance, susceptance and capacity) and generator design parameters (capacity and minimum power output) can be similarly computed.

The changes in LMPs as parameters vary provide insight on the functioning and behavior of the electric energy system. This sensitivity information might help producers and consumers to establish their respective bidding strategies, and the regulator to assess the degree of competitiveness of the electricity market. However, note that sensitivities provide information for small changes, not large changes.

Within an OPF market clearing framework, this paper provides expressions to compute the sensitivities of LMPs with respect to power demands. Sensitivities with respect to other parameters are also easily derived.

Sensitivity calculations reported in the literature are generally related to the power flow problem [11]-[14], or the optimal power flow problem involving sensitivities of the objective function and the primal variables with respect to parameters [15]. In this paper, within an OPF framework, we provide formulas to compute the sensitivities of dual variables (LMPs) with respect to parameters, that is, sensitivities of sensitivities.

In the appendix of pioneering conference paper [16], the sensitivities of certain primal variables (power injections) with respect to certain dual variables (LMPs) are calculated. The objective function is considered linear and only equality constraints are taken into account. A perturbation technique similar to the one used in this paper (and proposed in [17] in a mathematical programming framework) is used. Results reported in [16] are used in [18] and [19]. However, our paper calculates the sensitivities of any dual variable with respect to any parameter of the problem in a general nonlinear programming case including equality and inequality constraints.

The problem of sensitivity analysis in nonlinear programming has been discussed by several authors, as, for example, [20]–[27] and [17]. There are at least three ways of deriving equations for the sensitivities: (a) the Lagrange multiplier equations of the constrained optimum (see [20]), (b) differentiation of the Karush-Kuhn-Tucker conditions to obtain the sensitivities of the objective function with respect to changes in the parameters (see [22]), and (c) the extreme conditions of a penalty function (see [20]). In the derivations below, we use results reported in [17].

Background on electricity market can be found, for instance, in [28]-[30].

This paper is organized as follows. Section II provides the considered OPF formulation and defines the LMP sensitivities of interest. Section III develops analytical linear expression to compute LMP sensitivities. Section IV gives results from an illustrative example to demonstrate the functioning of the expressions derived. Section V gives results from a case study based on the IEEE Reliability Test System [31]. Section VI provides some relevant conclusions.

II. OPF Formulation and LMP Sensitivities

A general expression for the OPF has the form:

$$\underset{\boldsymbol{p}_G, \boldsymbol{q}_G, \boldsymbol{v}, \boldsymbol{\delta}}{\text{Minimize}} \quad z = f(\boldsymbol{p}_G, \boldsymbol{q}_G, \boldsymbol{v}, \boldsymbol{\delta}; \boldsymbol{p}_D, \boldsymbol{q}_D, \boldsymbol{c}) \tag{1}$$

subject to

$$p_{Gi} - p_{Di} - h_{Pi}(\boldsymbol{p}_G, \boldsymbol{q}_G, \boldsymbol{v}, \boldsymbol{\delta}; \boldsymbol{p}_D, \boldsymbol{q}_D, \boldsymbol{c}) = 0 : \lambda_i \quad i = 1, \ldots, n_B \tag{2}$$

$$q_{Gi} - q_{Di} - h_{Qi}(\boldsymbol{p}_G, \boldsymbol{q}_G, \boldsymbol{v}, \boldsymbol{\delta}; \boldsymbol{p}_D, \boldsymbol{q}_D, \boldsymbol{c}) = 0 \quad i = 1, \ldots, n_B \tag{3}$$

$$-(s_{ij}^{\max})^2 + (g_{Sij}(\boldsymbol{p}_G, \boldsymbol{q}_G, \boldsymbol{v}, \boldsymbol{\delta}; \boldsymbol{p}_D, \boldsymbol{q}_D, \boldsymbol{c}))^2 \leq 0 \quad i = 1, \ldots, n_B; \forall j \in \Omega_i \tag{4}$$

$$\boldsymbol{p}_G^{\min} - \boldsymbol{p}_G \leq \boldsymbol{0} \tag{5}$$

$$\boldsymbol{p}_G - \boldsymbol{p}_G^{\max} \leq \boldsymbol{0} \tag{6}$$

$$\boldsymbol{q}_G^{\min} - \boldsymbol{q}_G \leq \boldsymbol{0} \tag{7}$$

$$\boldsymbol{q}_G - \boldsymbol{q}_G^{\max} \leq \boldsymbol{0} \tag{8}$$

$$\boldsymbol{v}^{\min} - \boldsymbol{v} \leq \boldsymbol{0} \tag{9}$$

$$\boldsymbol{v} - \boldsymbol{v}^{\max} \leq \boldsymbol{0} \tag{10}$$

$$-\boldsymbol{\delta} - \boldsymbol{\pi} \leq \boldsymbol{0} \tag{11}$$

$$\boldsymbol{\delta} - \boldsymbol{\pi} \leq \boldsymbol{0} \tag{12}$$

Equation (1) represents generation cost (the minus social welfare in a market framework), equations (2)-(3) are the active and reactive power flow equations, respectively; constraints (4) enforce transmission capacity limits of power lines, constraints (5)-(6), (7)-(8), (9)-(10) and (11)-(12) are bounds on active power generations, reactive power generations, voltage magnitudes, and voltage angles, respectively.

For simplicity we consider all demands inelastic. Note, however, that elastic demand can be handled by the proposed procedure.

Particularly, we develop analytical expressions to compute the sensitivities

$$\frac{d\lambda_i}{dp_{Dj}} \tag{13}$$

for all $i, j = 1, \ldots, n_B$; that is, the sensitivity of any LMP with respect to the demand in any bus. Nevertheless, any other sensitivity can be obtained using the procedure described in Section III. In the case study, we also provide the following sensitivities:

$$\frac{d\lambda_i}{dq_{Dj}}, \quad \frac{d\lambda_i}{dv^{\max}}, \quad \frac{d\lambda_i}{da_j}, \quad \frac{d\lambda_i}{db_j} \tag{14}$$

where a_j and b_j are respectively the linear and quadratic cost coefficients of generator j.

To facilitate mathematical derivations, problem (1)-(12) is rewritten in compact form as:

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad z = f(\boldsymbol{x}, \boldsymbol{a}) \tag{15}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = \boldsymbol{0} : \boldsymbol{\lambda} \tag{16}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) \leq \boldsymbol{0} : \boldsymbol{\mu} \tag{17}$$

where variable vector $\boldsymbol{x} \in \mathbb{R}^n$, parameter vector $\boldsymbol{a} \in \mathbb{R}^p$, $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = (h_1(\boldsymbol{x}, \boldsymbol{a}), \ldots, h_\ell(\boldsymbol{x}, \boldsymbol{a}))^T$ and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) =$

$(g_1(\boldsymbol{x},\boldsymbol{a}),\ldots,g_m(\boldsymbol{x},\boldsymbol{a}))^T$ are the equality and inequality constraints, respectively. Vector $\boldsymbol{x}$ includes all optimization variables $(\boldsymbol{v},\boldsymbol{\delta},\boldsymbol{p}_G,\boldsymbol{q}_G)$, while vector $\boldsymbol{a}$ includes all parameters $(\boldsymbol{p}_D,\boldsymbol{q}_D,\boldsymbol{c})$; and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the Lagrange multiplier vectors for equality and inequality constraints, respectively. Parameter vector $\boldsymbol{c}$ includes line resistances, reactances, susceptances and capacities; generator capacities, minimum power outputs and cost coefficients; and voltage magnitude and angle limits. Note that l, m, n and p are the number of equality constraints, the number of inequality constraints, the number of variables, and the number of parameters, respectively.

III. General Sensitivity Expressions

A. Optimality Conditions

Considering appropriate regularity assumption[1] (see [32] or [33]) on problem (15)-(17), the Karush–Kuhn–Tucker (KKT) first order optimality conditions for this problem are

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \, \nabla_{\boldsymbol{x}} h_k(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{j=1}^{m} \mu_j^* \, \nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*,\boldsymbol{a}) = \boldsymbol{0} \tag{18}$$

$$h_k(\boldsymbol{x}^*,\boldsymbol{a}) = 0; \quad k=1,2,\ldots,\ell \tag{19}$$

$$g_j(\boldsymbol{x}^*,\boldsymbol{a}) \le 0; \quad j=1,2,\ldots,m \tag{20}$$

$$\mu_j^* \, g_j(\boldsymbol{x}^*,\boldsymbol{a}) = 0; \quad j=1,2,\ldots,m \tag{21}$$

$$\mu_j^* \ge 0; \quad j=1,2,\ldots,m \tag{22}$$

As it is well known, the vectors $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ are called the *Lagrange multipliers*. Conditions (19)–(20) are *the primal feasibility* conditions, condition (21) is known as the *complementary slackness condition*, and condition (22) imposes the nonnegativity of the multipliers of the inequality constraints, and is referred to as the *dual feasibility condition*.

The sensitivity analysis we propose requires an OPF solution. If such solution is not available, the analysis cannot be carried out. However, algorithms to solve the OPF in a robust manner are nowadays available. Moreover, if an OPF solution cannot be found, the OPF can be linearized, a solution obtained for this linearized OPF, and the proposed sensitivity analysis performed.

B. Feasible Perturbation

To obtain sensitivity equations, we perturb or modify $\boldsymbol{x}^*$, $\boldsymbol{a}$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, z^* in such a way that the KKT conditions still hold [17]. Thus, to obtain the sensitivity equations we differentiate the objective function (15) and the optimality conditions (18)-(22), as follows:

$$[\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*,\boldsymbol{a})]^T \, d\boldsymbol{x} + [\nabla_{\boldsymbol{a}} f(\boldsymbol{x}^*,\boldsymbol{a})]^T \, d\boldsymbol{a} - dz = 0 \tag{23}$$

$$\begin{aligned} &\left[\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*,\boldsymbol{a})\right] d\boldsymbol{x} \,+ \\ &\left[\nabla_{\boldsymbol{xa}} f(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xa}} h_k(\boldsymbol{x}^*,\boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xa}} g_j(\boldsymbol{x}^*,\boldsymbol{a})\right] d\boldsymbol{a} \,+ \\ &\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{a}) d\boldsymbol{\lambda} + \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{a}) d\boldsymbol{\mu} = \boldsymbol{0} \end{aligned} \tag{24}$$

$$[\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{a})]^T d\boldsymbol{x} + [\nabla_{\boldsymbol{a}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{a})]^T d\boldsymbol{a} = \boldsymbol{0} \tag{25}$$

$$[\nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*,\boldsymbol{a})]^T d\boldsymbol{x} + [\nabla_{\boldsymbol{a}} g_j(\boldsymbol{x}^*,\boldsymbol{a})]^T d\boldsymbol{a} = 0 \quad \text{if } \mu_j^* \neq 0; j \in J \tag{26}$$

where J is the set of binding (active) inequality constraints, m_J its cardinality, and all the matrices are evaluated at the optimal solution, $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, z^*. It should be noted that the derivation above is based on results reported in [17].

It should also be noted that once an OPF solution is known, binding inequality constraints are considered equality constraints and non-binding ones are disregarded. Note that this is appropriate as our analysis is just local. Note also that we assume local convexity around an optimal OPF solution.

Observe that the degenerate case (binding constraints with null multipliers) is not considered. Once the OPF solution is known, degeneracy can be identified and eliminated. The degenerate case is extensively analyzed in [17].

The linear system of equations (23)-(26) can be expressed in matrix form as follows

$$\left[\begin{array}{c|c|c|c|c} \boldsymbol{F_x} & \boldsymbol{F_a} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{F_{xa}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{H_a} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x} & \boldsymbol{G_a} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right] \left[\begin{array}{c} d\boldsymbol{x} \\ d\boldsymbol{a} \\ d\boldsymbol{\lambda} \\ d\boldsymbol{\mu} \\ dz \end{array}\right] = \boldsymbol{0} \tag{27}$$

where the vectors and submatrices in (27) are defined below

[1]For a given OPF solution, regularity entails that the gradient vectors of the binding constraints at the solution are linearly independent. Non regular cases are extensively analyzed in [17].

(dimensions in parenthesis)

$$\boldsymbol{F_x}_{(1\times n)} = [\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{a})]^T \quad (28)$$

$$\boldsymbol{F_a}_{(1\times p)} = (\nabla_{\boldsymbol{a}} f(\boldsymbol{x}^*, \boldsymbol{a}))^T \quad (29)$$

$$\boldsymbol{F_{xx}}_{(n\times n)} = \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \quad (30)$$

$$\boldsymbol{F_{xa}}_{(n\times p)} = \nabla_{\boldsymbol{xa}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xa}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xa}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \quad (31)$$

$$\boldsymbol{H_x}_{(\ell\times n)} = [\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a})]^T \quad (32)$$

$$\boldsymbol{H_a}_{(\ell\times p)} = [\nabla_{\boldsymbol{a}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a})]^T \quad (33)$$

$$\boldsymbol{G_x}_{(m_J\times n)} = [\nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a})]^T \quad (34)$$

$$\boldsymbol{G_a}_{(m_J\times p)} = [\nabla_{\boldsymbol{a}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a})]^T \ . \quad (35)$$

Vector (28) is the gradient of the objective function with respect to $\boldsymbol{x}$, vector (29) is the gradient on the objective function with respect to $\boldsymbol{a}$, submatrix (30) is the Hessian of the Lagrangian ($f(\boldsymbol{x},\boldsymbol{a})+\boldsymbol{\lambda}^T\boldsymbol{h}(\boldsymbol{x},\boldsymbol{a})+\boldsymbol{\mu}^T(\boldsymbol{x},\boldsymbol{a})$) with respect to $\boldsymbol{x}$, submatrix (31) is the Hessian of the Lagrangian with respect to $\boldsymbol{x}$ and $\boldsymbol{a}$, submatrix (32) is the Jacobian of $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{a})$ with respect to $\boldsymbol{x}$, submatrix (33) is the Jacobian of $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{a})$ with respect to $\boldsymbol{a}$, submatrix (34) is the Jacobian of $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{a})$ with respect to $\boldsymbol{x}$ for binding constraints, and submatrix (35) is the Jacobian of $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{a})$ with respect to $\boldsymbol{a}$ for binding constraints.

C. Specific Sensitivity Expressions

To compute sensitivities with respect to the components of the parameter vector $\boldsymbol{a}$, system (27) can be written as

$$\boldsymbol{U}\,[\ d\boldsymbol{x} \quad d\boldsymbol{\lambda} \quad d\boldsymbol{\mu} \quad dz\ ]^T = \boldsymbol{S}d\boldsymbol{a} \quad (36)$$

where the matrices $\boldsymbol{U}$ and $\boldsymbol{S}$ are

$$\boldsymbol{U} = \left[\begin{array}{c|c|c|c} \boldsymbol{F_x} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right] \quad (37)$$

$$\boldsymbol{S}^T = -[\ \boldsymbol{F_a} \quad \boldsymbol{F_{xa}} \quad \boldsymbol{H_a} \quad \boldsymbol{G_a}\ ] \quad (38)$$

and therefore

$$\left[\ d\boldsymbol{x} \quad d\boldsymbol{\lambda} \quad d\boldsymbol{\mu} \quad dz\ \right]^T = \boldsymbol{U}^{-1}\boldsymbol{S}\,d\boldsymbol{a} \ . \quad (39)$$

Replacing $d\boldsymbol{a}$ by the p-dimensional identity matrix $\boldsymbol{I}$ in (39) all the derivatives are obtained. Thus, the matrix with all derivatives with respect to parameters becomes

$$\left[\ \frac{d\boldsymbol{x}}{d\boldsymbol{a}} \quad \frac{d\boldsymbol{\lambda}}{d\boldsymbol{a}} \quad \frac{d\boldsymbol{\mu}}{d\boldsymbol{a}} \quad \frac{dz}{d\boldsymbol{a}}\ \right]^T = \boldsymbol{U}^{-1}\boldsymbol{S} \ . \quad (40)$$

Expression (40) allows deriving sensitivities of the variables, the multipliers (dual variables) and the objective function with respect to all parameters. Therefore, the sensitivities of the LMPs with respect to active and reactive power demands are straightforwardly obtained using expression (40). The simplicity of expression (40) should be noted.

The computational complexity of building matrices $\boldsymbol{U}$ and $\boldsymbol{S}$ and evaluating expression (40) is moderate even for large scale electric energy systems.

It should be noted that matrix $\boldsymbol{U}$ is invertible in most practical cases, e. g. if the solution is regular and non-degenerate. However, if it is not, alternative procedures (more computationally involved) to obtain and/or analyze the sensitivities are available [17].

For linear programming problems the analysis remains valid. Matrix U is invertible provided that (a) only binding constrains are considered, (b) redundant constraints are removed and (c) non-basic variables are eliminated. This implies that the constraint matrix of the linear programming problem reduces to the basis. Note finally that the basis is known once the linear programming problem has been solved.

IV. Illustrative Example

The 6-bus electric energy system depicted in Fig. 1 is considered in this example [29]. Data for this system is provided in the Appendix.

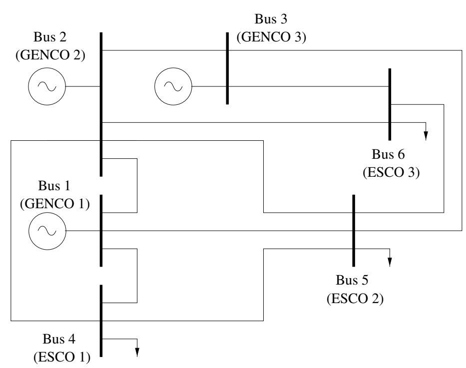

Fig. 1. Six-bus system.

The considered OPF formulation is stated below.

$$\underset{p_{Gi}, q_{Gi};\ i=1,\ldots,3;\quad v_j, \delta_j;\ j=1,\ldots,6}{\text{Minimize}} \quad \sum_{i=1}^{3} \left(a_i p_{Gi} + b_i p_{Gi}^2\right)$$

subject to

$$p_{Gi} - p_{Di} = v_i \sum_{i=1}^{6} v_j \left(G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)\right) \ : \lambda_i \ ; \quad i = 1, \ldots, 6$$

$$q_{Gi} - q_{Di} = v_i \sum_{i=1}^{6} v_j \left(G_{ij} \sin(\delta_i - \delta_j) - B_{ij} \cos(\delta_i - \delta_j)\right) \ ; \quad i = 1, \ldots, 6$$

$$\left(v_i v_j (G_{ij} \cos(\delta_i - \delta_j) + B_{ij} \sin(\delta_i - \delta_j)) - G_{ij} v_i^2\right)^2 + \left(v_i v_j (G_{ij} \sin(\delta_i - \delta_j) - B_{ij} \cos(\delta_i - \delta_j)) - v_i^2 (B_{ij} - B_{ij}^S/2)\right)^2 \leq (s_{ij}^{\max})^2 \ ; \quad i = 1, \ldots, 6; j \in \Omega_i$$

$$\begin{aligned} & p_{Gi}^{\min} \leq p_{Gi} \leq p_{Gi}^{\max} \ ; \qquad i = 1, \cdots, 3 \\ & q_{Gi}^{\min} \leq q_{Gi} \leq q_{Gi}^{\max} \ ; \qquad i = 1, \cdots, 3 \\ & v^{\min} \leq v_i \leq v^{\max} \ ; \qquad i = 1, \cdots, 6 \\ & -\pi \leq \delta_i \leq \pi \ ; \qquad i = 2, \cdots, 6 \\ & \delta_1 = 0 \end{aligned}$$

where a_i, b_i, G_{ij}, B_{ij}, B_{ij}^S and $s_{ij}^{\max}$ (constants included in $\boldsymbol{c}$) are the linear cost coefficient of generator i, the quadratic cost coefficient of generator i, the element ij of the real part of the admittance matrix, the element ij of the imaginary part of the admittance matrix, the charging susceptance of line ij, and the transmission capacity of line ij, respectively. Additionally, it is assumed that $v^{\min} = 0.9$; and $v^{\max} = 1.1$.

The optimal solution of the OPF above is illustrated in Table I.

TABLE I
OPF SOLUTION

Bus #	p_{Gi}^* [MW]	q_{Gi}^* [MVAr]	v_i^* [p.u.]	δ_i^* [rad]	λ_{Pi}^* [\$/MWh]
1	132.5	37.3	1.100	0.000	8.977
2	160.6	92.9	1.100	-0.047	9.161
3	60.0	82.8	1.098	-0.091	9.430
4	-	-	1.018	-0.090	9.733
5	-	-	1.006	-0.121	9.866
6	-	-	1.034	-0.128	9.711

Sensitivities of LMPs with respect to active power demands are provided in the matrix below. Units are $(\$/\text{MWh})/(\text{puMW})$. If the desired units are $(\$/\text{MWh})/(\text{MW})$, the matrix below should be divided by 100, the power base value.

$$\left[\frac{d\lambda_i}{dp_{Dj}}\right] = \begin{bmatrix} 2.162 & 0.098 & 0.492 & 3.852 & 1.610 & 0.639 \\ 0.098 & 0.100 & 0.103 & 0.106 & 0.108 & 0.106 \\ 0.492 & 0.103 & 0.843 & 1.023 & 0.412 & 0.644 \\ 3.852 & 0.106 & 1.023 & 9.014 & 3.271 & 1.327 \\ 1.610 & 0.108 & 0.412 & 3.271 & 2.124 & 0.701 \\ 0.639 & 0.106 & 0.644 & 1.327 & 0.701 & 0.847 \end{bmatrix}$$

Note that in the expression above subscript i refers to rows while subscript j refers to columns. Note also that the matrix above is full as all LMPs change as any load increment occurs (in any bus throughout the network).

The observations below are pertinent:

1) Matrix $\left[\frac{d\lambda_i}{dp_{Dj}}\right]$ is symmetrical. This is a consequence of the linearity of the analysis carried out and the regularity of problem (15)-(17).
2) Being generator 2 the marginal generator, values in column/row 2 are small and close to 0.1. The actual values are related to the quadratic cost terms and the fluctuations (around 0.1) are due to the losses.
3) The derivative of the LMP in bus 2 with respect to the demand in that bus equals the derivative with respect to demand of the marginal cost of the generator in that bus, because this generator is the swing generator.
4) The highest price sensitivity occurs in bus 4. This indicates that this bus might suffer a comparatively significant price volatility. Observe that the high price volatility of bus 4 is not a trivial property of the system, thus demonstrating the added value of the proposed analysis.

It should be noted that if cost functions are linear, as is often assumed, the derivatives with respect to demand at generator swing buses are zero.

Sensitivities of LMPs with respect to reactive power demands are provided in the matrix below. Units are $(\$/\text{MWh})/(\text{puMVAr})$.

$$\left[\frac{d\lambda_i}{dq_{D_j}}\right] = \begin{bmatrix} 0 & 0 & 0 & 2.135 & 0.666 & 0.170 \\ 0 & 0 & 0 & 0.005 & 0.005 & 0.003 \\ 0 & 0 & 0 & 0.551 & -0.091 & -0.040 \\ 0 & 0 & 0 & 5.215 & 1.530 & 0.379 \\ 0 & 0 & 0 & 1.910 & 1.033 & 0.293 \\ 0 & 0 & 0 & 0.750 & 0.188 & 0.076 \end{bmatrix}$$

Observe that these sensitivities are much smaller than the sensitivities with respect to active power demands.

Note that the first three columns of the matrix of LMP derivatives with respect reactive power demands are zero because reactive power has no cost (in this particular example) and is in between its bounds for each of the generators at the first three buses.

Sensitivities with respect to the voltage single upper bound are provided by the vector below. Sensitivities with respect to the voltage single lower bound are zero, as no lower bound limit is reached for voltages (see Table I). Units are $(\$/\text{MWh})/(\text{puV})$.

$$\left[\frac{d\lambda_i}{dv^{\max}}\right] = \begin{bmatrix} -1.758 \\ -0.034 \\ -1.041 \\ -6.501 \\ -3.761 \\ -1.941 \end{bmatrix}$$

Observe that bus 4 presents the highest sensitivities with respect to the reactive power demands and with respect to the single upper bound for voltage magnitudes, which leads to the conclusion that the demand in bus 4 plays a critical role in the congestion of the network and confirms the volatility of λ_4.

Next, sensitivities with respect to generator (linear and quadratic) cost parameters a_j and b_j are provided in the matrices below

$$\left[\frac{d\lambda_i}{da_j}\right] = \begin{bmatrix} 0 & 0.980 & 0 \\ 0 & 1.000 & 0 \\ 0 & 1.029 & 0 \\ 0 & 1.063 & 0 \\ 0 & 1.077 & 0 \\ 0 & 1.060 & 0 \end{bmatrix}$$

and

$$\left[\frac{d\lambda_i}{db_j}\right] = \begin{bmatrix} 0 & 314.9 & 0 \\ 0 & 321.4 & 0 \\ 0 & 330.8 & 0 \\ 0 & 341.4 & 0 \\ 0 & 346.1 & 0 \\ 0 & 340.6 & 0 \end{bmatrix}$$

The first matrix above is dimensionless and units for the second matrix are $(\$/\mathrm{MWh})/(\$/(\mathrm{MW})^2\mathrm{h})$.

Observe that sensitivities of LMPs with respect to cost coefficients are zero for generators working at their respective maximum or minimum power output.

Note also that the derivative of the LMP in bus 2 with respect to the linear cost term of the generator in that bus is 1 because that generator is the swing generator.

Finally, note that sensitivities with respect to line design parameters (resistance, reactance, susceptance and capacity) and generator design parameters (capacity, minimum power output and cost parameters) are also readily available.

V. Case Study

A case study based on the IEEE RTS, depicted in Fig. 2, is presented in this section. Topology, line and generator data can be found in [31] (Fig. 1 and Tables 12 and 9, respectively, of that reference). Fuel costs have been taken from [34] and are \$2.3/MBtu for #6 oil, \$3.0/MBtu for #2 oil, \$1.20/MBtu for coal, and \$0.6/MBtu for nuclear.

The methodology presented in the previous sections is applied to the IEEE Reliability Test System in this section. A selection of the results obtained is described and discussed below. Fig. 3 provides the LMP sensitivities of bus 21 (a typical generating bus) with respect to active and reactive power demands, respectively; while Fig. 4 provides the same information for bus 8 (a typical demand bus). Units for these sensitivities are $(\$/\mathrm{MWh})/(\mathrm{puMW})$ and $(\$/\mathrm{MWh})/(\mathrm{puMVAr})$, respectively.

As expected, sensitivities of LMPs with respect to active power variations are significantly higher that sensitivities with respect to reactive power increments.

It should be noted that the highest values of LMP sensitivities with respect to active power demand variations are the diagonal elements (i.e. $d\lambda_i/dp_{D_i}$) of the corresponding sensitivity matrix. Non-diagonal sensitivities depend directly on the transmission system topology and parameters, and can be used as a measure of the impact of demand variations in any bus throughout the power system on the network working conditions. For example, LMP λ_{21} presents high sensitivities with respect to demand variations in its adjacent buses, while the sensitivity is relatively low for a demand variation in bus 7, which is electrically far away from bus 21. On the other hand, LMP λ_8 presents a relatively significant sensitivity with respect to the demand variation in adjacent bus 7, while suffering a homogeneous and relativity weak impact from demand variation in the remaining buses.

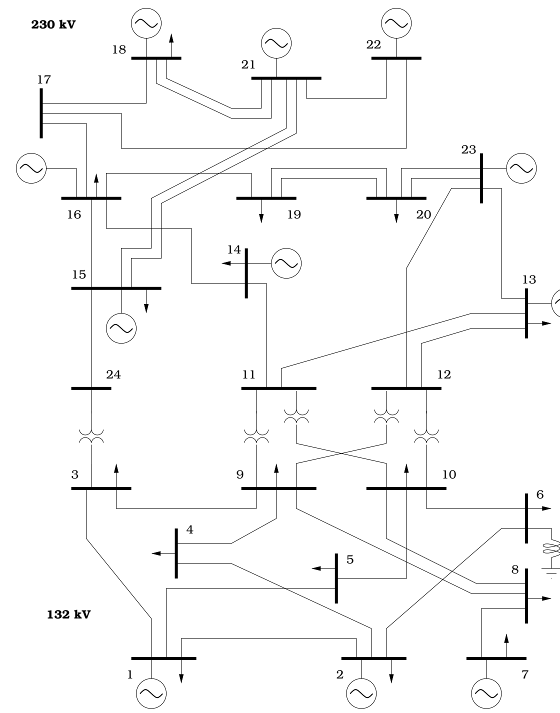

Fig. 2. IEEE 24-Bus Reliability Test System [31].

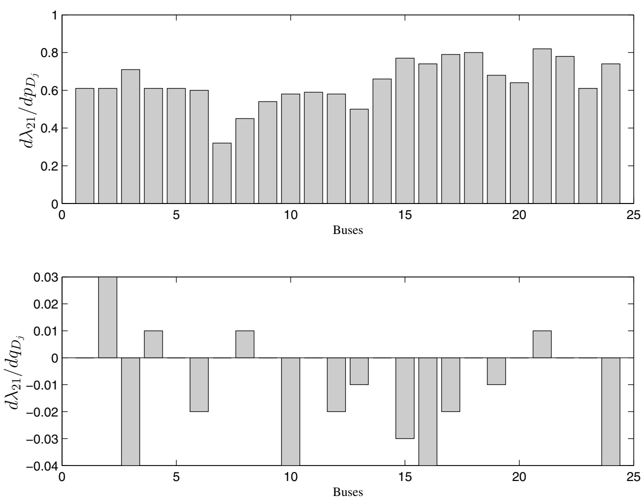

Fig. 3. Sensitivities $d\lambda_{21}/dp_{D_j}$ and $d\lambda_{21}/dq_{D_j}$ for the 24-bus test system.

VI. Conclusion

This paper provides simple analytical expressions to compute LMP sensitivities with respect to changes in demands throughout an electric power network. Not only prices but their sensitivities with respect to demands constitute fundamental

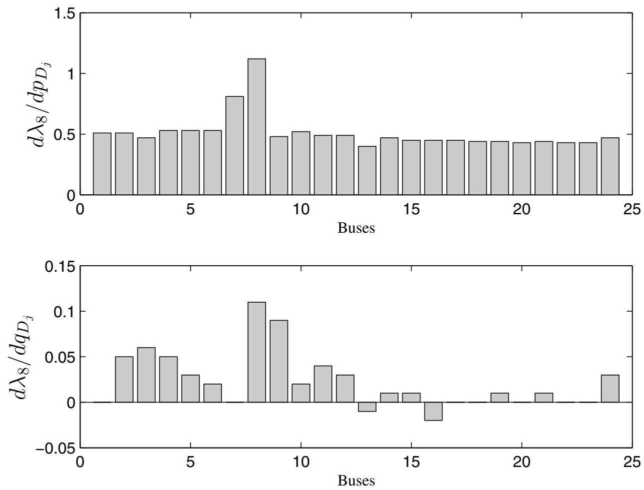

Fig. 4. Sensitivities $d\lambda_8/dp_{D_j}$ and $d\lambda_8/dq_{D_j}$ for the 24-bus test system.

information in nowadays mature electricity markets. An example and a case study are used to illustrate the sensitivity formulas derived.

Appendix
Data for the Six-Bus System

Generator, demand and line data are provided in Tables II, III and IV, respectively. As it is customary, the considered three-phase power base is 100 MVA.

TABLE II
GENERATOR DATA

Bus #	$p_{Gi}^{\max}$ [MW]	$p_{Gi}^{\min}$ [MW]	$q_{Gi}^{\max}$ [MVAr]	$q_{Gi}^{\min}$ [MVAr]	a_i [\$/MWh]	b_i [\$/(MW)^{2}h]
1	132.5	112.5	150.0	-150.0	8.5	0.0005
2	165.0	140.0	150.0	-150.0	9.0	0.0005
3	80.0	60.0	150.0	-150.0	9.5	0.0005

TABLE III
DEMAND DATA

Bus #	p_{Di} [MW]	q_{Di} [MVAr]
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	120.0	80.0
5	115.0	82.0
6	104.0	66.0

TABLE IV
LINE DATA

Line #	From To	R_{ij} [p.u.]	X_{ij} [p.u.]	B_{ij}^S [p.u.]	$s_{ij}^{\max}$ [MVA]
1	1-2	0.10	0.20	0.04	36.0
2	1-4	0.05	0.20	0.04	72.0
3	1-5	0.08	0.30	0.06	63.6
4	2-3	0.05	0.25	0.06	36.0
5	2-4	0.05	0.10	0.02	91.2
6	2-5	0.10	0.30	0.04	42.0
7	2-6	0.07	0.20	0.05	72.0
8	3-5	0.12	0.26	0.05	36.0
9	3-6	0.02	0.10	0.02	84.0
10	4-5	0.2	0.40	0.08	18.0
11	5-6	0.10	0.30	0.06	14.4

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