Interpreting Systems of Equalities and Inequalities.

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Contents

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Abstract

This paper shows how the mathematical and the engineering points of view are complementary and help to model real problems that can be stated as systems of linear equations and inequalities. The paper is devoted to point out these relations and making them explicit for the readers to discover the new world that arises when contemplating the compatibility conditions or the set of general solutions from the dual perspective. After reminding an orthogonally based powerful algorithm to analyze the compatibility of linear systems of equations and solving them, a water supply problem is used to illustrate its mathematical and engineering multiple aspects, including the optimal statement of the problem in terms of an adequate selection and numbering of equations and unknowns, a deep analysis of the compatibility conditions and a physical interpretation of the general solution, together with that of each individual generators of the affine space. The possibilities of removing unknowns without altering the compatibility of the problem is also analyzed. Next, the Γ-algorithm to analyze the compatibility of linear systems of inequalities and solving them is described and then, the water supply problem revisited adding some constraints, such as capacity limits for the pipes and retention valves, and discussing how they affect the resulting general solution and many other aspects of it. Finally, some conclusions are derived.

Key Words: Compatibility, cones, duality, linear spaces, polytopes, simultaneous solutions .

1 Introduction

There are many physical and engineering problems that involve linear systems of equalities and inequalities. These systems can be interpreted from the mathematical or the engineering points of view, that are complementary and terribly rich. In general, people working in these areas have knowledge about only one of these two perspectives and lack a deep understanding of the relations between the mathematical and the physical concepts. This fact, leads to important limitations in the capacity of extracting conclusions from the results that can be expected after a careful analysis of these problems.

This paper is devoted to pointing out these relations and making them explicit for the readers to discover the new world that arises when contemplating the compatibility conditions or the set of general solutions from the dual perspective.

Though many other possibilities exist, we have selected a particular example, the water supply problem, to illustrate these two points of view, and we exploit this dual (mathematical and engineering) perspective to deal with a problem that involves linear systems of equalities or inequalities, depending on the constraints used to model the reality. Many questions of practical interest arise and can be answered thanks to this dual analysis of the problem.

For the sake of completeness two algorithms, one for dealing with linear equalities and one for linear inequalities are given. They allow solving not only particular systems of equations but many subsets at the same time.

The paper is structured as follows. In Section 2 we describe one of the two fundamental algorithms in this paper, the orthogonalization algorithm, that is used in Section 3 for determining the compatibility of systems of linear equalities and solving them, i.e., obtaining the set of all possible solutions. In Section 4 the water supply problem is described and used to illustrate all the theoretical methods. A special care is taken in showing the correspondences between the engineering and the mathematical elements of this problem. Section 5 presents the Γ-algorithm, a powerful algorithm to obtain the dual cone of a given cone, that is the key algorithm used in Section 6 for determining the compatibility of systems of linear inequalities and solving them, i.e., obtaining the set of all possible solutions. In Section 7 we revisit the water supply problem adding some constraints to illustrate the methodology and several engineering problems and the correspondences with their mathematical counterparts discussed. Finally, in Section 8 we give some conclusions and recommendations.

2 The Orthogonalization Algorithm

In this section we describe an important algorithm for obtaining the linear orthogonal subspace orthogonal to another linear subspace (for a detailed description of this algorithm see Castillo, Cobo, Jubete, Pruneda and Castillo [3], Castillo, Cobo, Fernández-Canteli, Jubete and Pruneda [1], Cobo, Jubete and Pruneda [2]). We shall see that this algorithm allows studying the compatibility and solving linear systems of equalities.

Algorithm 1 (Orthogonalization Algorithm)

- **Input:** A set of vectors $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$ of $\mathbb{R}^n$.
- **Output:** The generators $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$ of the linear subspace $\mathcal{L}(\mathbf{V})^\perp$ orthogonal to $\mathcal{L}(\mathbf{V})$, where $\mathbf{V}$ is any subset of $\mathbf{U}$, and the list L of pivot columns, necessary to obtain the generators of a given $\mathbf{V}$.

Step 1: Set $\mathbf{W} = \mathbf{I}_n$ (the identity matrix of dimension n and let the iteration number $i = 1$.

Step 2: Calculate the dot products $t_j^i = \mathbf{u}_i^T \mathbf{w}_j$ for all j, where $\mathbf{w}_j$ is the column j vector of $\mathbf{W}$.

Step 3: Choose the pivot column p as one column vector not orthogonal to $\mathbf{u}_i$, that is, $t_p^i \neq 0$ and add it to the pivot list L. If there is no such a column go to Step 6. Otherwise, continue with Step 4.

Step 4: Divide the actual pivot column $\mathbf{w}_p$ by t_p.

Step 5: For all the remaining columns (different from the actual pivot column, i.e., $j \neq p$) do $\mathbf{w}_j = \mathbf{w}_j - t_j^i \mathbf{w}_p$.

Step 6: If $i = m$, go to Step 7. Otherwise, increase i in one unit and go to Step 2.

Step 7: Return $\mathbf{W} = \{\mathbf{w}_\ell, \ldots, \mathbf{w}_s\}$ as a set containing the generators of all linear subspaces orthogonal to all possible $\mathcal{L}(\mathbf{V})$, and the list L of pivot columns.

Remark 1 *After each iteration of the algorithm, i.e., after Step 5, each column in* $\mathbf{W}$ *can be multiplied by an arbitrary non-null number without altering the validity of the algorithm.*

Remark 2 *Given* $\mathbf{V}$*, to obtain the linear subspace orthogonal to it, we remove from* $\mathbf{W}$ *the pivot columns associated with the vectors in* $\mathbf{V}$ *and return the linear subspace generated by the remaining columns. The complement linear subspace can be obtained from the pivot columns.*

3 Solving Systems of Equations

This section is devoted to systems of equations of the form:

$$\begin{array}{ccccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = & b_1, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = & b_2, \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

First, a method for determining whether or not the system (1) of equations is compatible is given. Next, it is shown how the set of all possible solutions of the given system can be obtained.

A classical treatment of these problems can be seen in Gill, Golub, Murray and Saunders [10].

3.1 Deciding whether or not a linear system of equations is compatible

In this section we show how to apply the orthogonalization algorithm to analyze the compatibility of a given system of equations.

System (1) can be written as

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{2}$$

Expression (2) shows that the vector $\mathbf{b} = (b_1, \dots, b_m)^T$ belongs the linear space generated by the column vectors $\{\mathbf{a}_1, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ of the system matrix $\mathbf{A}$, i.e., the compatibility requires:

$$\mathbf{b} \in \mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) \Leftrightarrow \mathbf{b} \in \left(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)^{\perp}\right)^{\perp}.$$

Thus, analyzing the compatibility of the system of equations (1) reduces to finding the linear subspace $\mathcal{L}\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ orthogonal to $\mathcal{L}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and checking whether or not $\mathbf{b}^T\mathbf{W} = \mathbf{0}$.

Example 1 (Compatibility of a linear system of equations) *Suppose that we are interested in determining the conditions under which the system of equations*

$$\begin{array}{rrrrcr} -x_1 & +x_2 & & +2x_4 & = & a \\ 2x_1 & +x_2 & -x_3 & -x_4 & = & 2a \\ & & -x_3 & +x_4 & = & -a \\ & x_2 & +x_3 & +x_4 & = & b \\ 2x_1 & -2x_2 & +x_3 & & = & c \end{array} \tag{3}$$

is compatible. Then, using Algorithm 1, we get (see Table 1) that the linear subspace orthogonal to the linear subspace generated by the column vectors in (3) is:

$$\mathbf{W} = \mathcal{L}\{\mathbf{w}_1\} = \mathcal{L}\left\{(18, 5, -16, -15, 4)^T\right\}, \tag{4}$$

which implies the following compatibility condition:

$$\mathbf{w}_1^T(a, 2a, -a, b, c)^T \quad = \quad (18, 5, -16, -15, 4)(a, 2a, -a, b, c)^T = 0 \Rightarrow 44a - 15b + 4c = 0. \tag{5}$$

3.2 Solving a homogeneous system of linear equations

Consider the homogeneous system of equations

$$\begin{array}{ccccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = & 0, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = & 0, \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = & 0 \end{array} \tag{6}$$

which can be written as

$$\begin{array}{rcl} (a_{11}, \dots, a_{1n})(x_1, \dots, x_n)^T & = & 0, \\ (a_{21}, \dots, a_{2n})(x_1, \dots, x_n)^T & = & 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots \\ (a_{m1}, \dots, a_{mn})(x_1, \dots, x_n)^T & = & 0. \end{array} \tag{7}$$

Expression (7) shows that $(x_1, \dots, x_n)$ is orthogonal to the set of row vectors $\{\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^m\}$ of $\mathbf{A}$.

Then, obtaining the solution to system (7) reduces to determining the linear subspace orthogonal to the linear subspace generated by the rows of matrix $\mathbf{A}$.

Iteration 1

$\mathbf{a}_1$	$\mathbf{v}_1^1$	$\mathbf{v}_2^1$	$\mathbf{v}_3^1$	$\mathbf{v}_4^1$	$\mathbf{v}_5^1$
-1	**1**	0	0	0	0
2	**0**	1	0	0	0
0	**0**	0	1	0	0
0	**0**	0	0	1	0
2	**0**	0	0	0	1
$\mathbf{t}^1$	**-1**	2	0	0	2

Iteration 2

$\mathbf{a}_2$	$\mathbf{v}_1^2$	$\mathbf{v}_2^2$	$\mathbf{v}_3^2$	$\mathbf{v}_4^2$	$\mathbf{v}_5^2$
1	-1	**2**	0	0	2
1	0	**1**	0	0	0
0	0	**0**	1	0	0
1	0	**0**	0	1	0
-2	0	**0**	0	0	1
$\mathbf{t}^2$	-1	**3**	0	1	0

Iteration 3

$\mathbf{v}_1^3$	$\mathbf{v}_2^3$	$\mathbf{v}_3^3$	$\mathbf{v}_4^3$	$\mathbf{v}_5^3$
-1/3	2/3	0	-2/3	2
1/3	1/3	0	-1/3	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Iteration 3 (simplified)

$\mathbf{a}_3$	$\mathbf{v}_1^3$	$\mathbf{v}_2^3$	$\mathbf{v}_3^3$	$\mathbf{v}_4^3$	$\mathbf{v}_5^3$
0	-1	2	**0**	-2	2
-1	1	1	**0**	-1	0
-1	0	0	**1**	0	0
1	0	0	**0**	3	0
1	0	0	**0**	0	1
$\mathbf{t}^3$	-1	-1	**-1**	4	1

Iteration 4

$\mathbf{a}_4$	$\mathbf{v}_1^4$	$\mathbf{v}_2^4$	$\mathbf{v}_3^4$	$\mathbf{v}_4^4$	$\mathbf{v}_5^4$
2	-1	2	0	**-2**	2
-1	1	1	0	**-1**	0
1	-1	-1	-1	**4**	1
1	0	0	0	**3**	0
0	0	0	0	**0**	1
$\mathbf{t}^4$	-4	2	-1	**4**	5

Final

$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
-3	3	-1/2	-1/2	9/2
0	3/2	-1/4	-1/4	5/4
3	-3	0	1	-4
3	-3/2	3/4	3/4	-15/4
0	0	0	0	1

Final (simplified)

$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
-1	2	-2	-2	18
0	1	-1	-1	5
1	-2	0	4	-16
1	-1	3	3	-15
0	0	0	0	4

Table 1: Tables resulting from the orthogonalization algorithm. Note that the tables in iteration 3 and the final tables have been simplified by multiplying some columns by non-null numbers.

Example 2 (Solving homogeneous systems of linear equations) *Consider the set of equations*

$$\begin{array}{rrrrrcl} -x_1 & +2x_2 & & & +2x_5 & = & 0 \\ x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ & -x_2 & -x_3 & +x_4 & +x_5 & = & 0 \\ 2x_1 & -x_2 & +x_3 & +x_4 & & = & 0. \end{array} \tag{8}$$

To solve any subsystem of (8), we obtain the generators of the linear subspaces orthogonal to any of the linear subspaces generated by the rows of the system matrix, as shown in Table 1, Iterations 1 to 4. Next, given a subset of equations, we remove the corresponding pivot columns from the final table and generate the orthogonal linear subspace with the remaining columns. In fact, we can solve $2^4 - 1 = 15$ *subsystems from this table. Some of them together with their solutions are given in Table 2.*

3.3 Solving a complete system of linear equations

Now consider the complete system of linear equations (1):

$$\begin{array}{rrrrcl} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = & b_1, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = & b_2, \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = & b_m \end{array} \tag{9}$$

that adding the artificial variable x_{n+1}, it can be written as

$$\begin{array}{rrrrrcl} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & -b_1x_{n+1} & = & 0, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & -b_2x_{n+1} & = & 0, \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & -b_mx_{n+1} & = & 0, \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & -b_mx_{n+1} & = & 0, \\ & & & & x_{n+1} & = & 1. \end{array} \tag{10}$$

Table 2: Some examples of homogeneous subsystems of (8) that can be solved from the final simplified tableau in Table 1.

System	Solution
$-x_1 + 2x_2 + 2x_5 = 0$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -2 & 18 \\ 1 & -1 & -1 & 5 \\ -2 & 0 & 4 & -16 \\ -1 & 3 & 3 & -15 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix}$
$-x_2 - x_3 + x_4 + x_5 = 0$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -2 & 18 \\ 0 & 1 & -1 & 5 \\ 1 & -2 & 4 & -16 \\ 1 & -1 & 3 & -15 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix}$
$\begin{array}{rrrrrcl} x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ & -x_2 & -x_3 & +x_4 & +x_5 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 18 \\ 0 & -1 & 5 \\ 1 & 4 & -16 \\ 1 & 3 & -15 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$
$\begin{array}{rrrrrcl} x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ 2x_1 & -x_2 & +x_3 & +x_4 & & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 18 \\ 0 & -1 & 5 \\ 1 & 0 & -16 \\ 1 & 3 & -15 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$
$\begin{array}{rrrrrcl} -x_1 & +2x_2 & & & +2x_5 & = & 0 \\ x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ 2x_1 & -x_2 & +x_3 & +x_4 & & = & 0. \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 & 18 \\ -1 & 5 \\ 0 & -16 \\ 3 & -15 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$
$\begin{array}{rrrrrcl} x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ & -x_2 & -x_3 & +x_4 & +x_5 & = & 0 \\ 2x_1 & -x_2 & +x_3 & +x_4 & & = & 0. \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 & 18 \\ 0 & 5 \\ 1 & -16 \\ 1 & -15 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$
$\begin{array}{rrrrrcl} -x_1 & +2x_2 & & & +2x_5 & = & 0 \\ x_1 & +x_2 & & +x_4 & -2x_5 & = & 0 \\ & -x_2 & -x_3 & +x_4 & +x_5 & = & 0 \\ 2x_1 & -x_2 & +x_3 & +x_4 & & = & 0. \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \begin{pmatrix} 18 \\ 5 \\ -16 \\ -15 \\ 4 \end{pmatrix}$

The first part of the system (10) can be written as

$$\begin{array}{rcl} (a_{11}, \ldots, a_{1n}, -b_1)(x_1, \ldots, x_n, x_{n+1})^T & = & 0, \\ (a_{21}, \ldots, a_{2n}, -b_2)(x_1, \ldots, x_n, x_{n+1})^T & = & 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \ldots \\ (a_{m1}, \ldots, a_{mn}, -b_m)(x_1, \ldots, x_n, x_{n+1})^T & = & 0. \end{array} \tag{11}$$

that shows that $(x_1, \ldots, x_n, x_{n+1})$ is orthogonal to the set of vectors

$$\{(a_{11}, \ldots, a_{1n}, -b_1), (a_{21}, \ldots, a_{2n}, -b_2), \ldots, (a_{m1}, \ldots, a_{mn}, -b_m)\}.$$

Then, it is clear that the solution of (11) is the linear subspace orthogonal to the linear subspace generated by the rows of matrix $\mathbf{A}_b$:

$$\mathcal{L}\{(a_{11}, \ldots, a_{1n}, -b_1), (a_{21}, \ldots, a_{2n}, -b_2), \ldots, (a_{m1}, \ldots, a_{mn}, -b_m)\}^{\perp}.$$

Final (simplified)				
$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
-1	*2*	*-2*	*-2*	9/2
0	*1*	*-1*	*-1*	5/4
1	*-2*	*0*	*4*	-4
1	*-1*	*3*	*3*	-15/4
0	*0*	*0*	*0*	1

Table 3: The final Table of the orthogonalization algorithm containing the generators of the solution of any subset of equations in (12).

Thus, the solution of (9) is the projection on $X_1 \times \cdots \times X_n$ of the intersection of the orthogonal complement of the linear subspace generated by

$$\{(a_{11}, \ldots, a_{1n}, -b_1), (a_{21}, \ldots, a_{2n}, -b_2), \ldots, (a_{m1}, \ldots, a_{mn}, -b_m)\}$$

and the set $\{\mathbf{x} | x_{n+1} = 1\}$.

Example 3 (A complete system of linear equations) *Consider the set of equations*

$$\begin{array}{rrrrcr} -x_1 & +2x_2 & & & = & -2 \\ x_1 & +x_2 & & +x_4 & = & 2 \\ & -x_2 & -x_3 & +x_4 & = & -1 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 0. \end{array} \tag{12}$$

which, using the auxiliary variable x_5, can be written as (8). Since the solution of the homogeneous system (8) was already obtained, now we only need to force $x_5 = 1$ and return to the initial set of variables (i.e., removing variable x_5) . For convenience we have included another version of the final tableau in Table 1 in Table 3. This tableau allows solving any subset of equations in (12). Table 4 shows some examples.

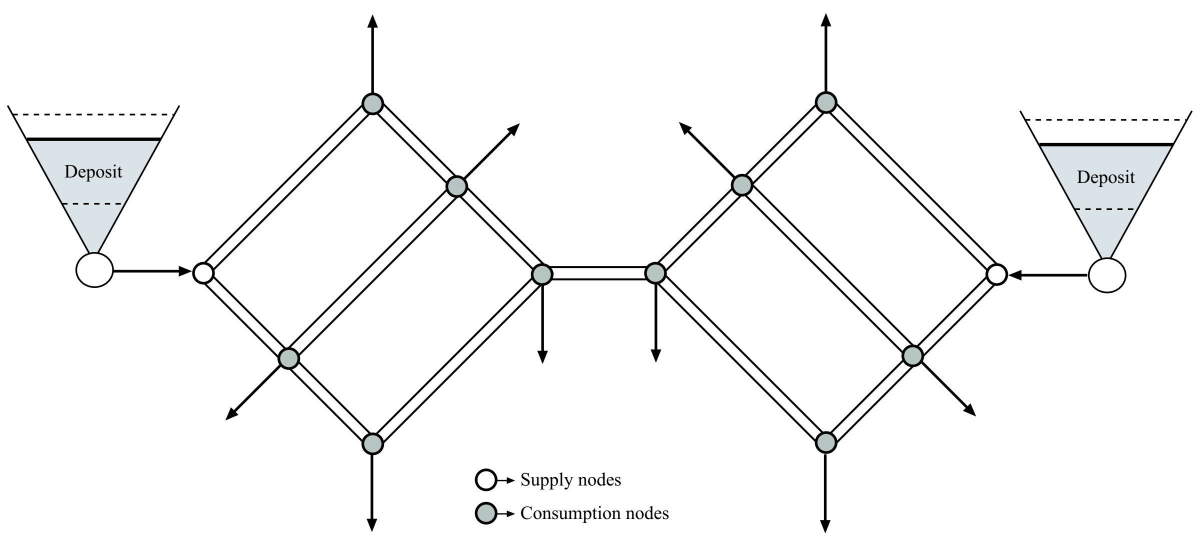

Figure 1: The water supply network showing the pipes, the nodes and the sign of the flows entering or leaving the network.

Table 4: Some examples of complete subsystems of (12) that can be solved from the final simplified tableau in Table 1.

System	Solution
$-x_1 + 2x_2 = -2$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -2 \\ 1 & -1 & -1 \\ -2 & 0 & 4 \\ -1 & 3 & 3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$-x_2 - x_3 + x_4 = -1$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$\begin{array}{rrrrcr} x_1 & +x_2 & & +x_4 & = & 2 \\ & -x_2 & -x_3 & +x_4 & = & -1 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \\ 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$\begin{array}{rrrrcr} x_1 & +x_2 & & +x_4 & = & 2 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$\begin{array}{rrrrcr} -x_1 & +2x_2 & & & = & -2 \\ x_1 & +x_2 & & +x_4 & = & 2 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 0. \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -2 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$\begin{array}{rrrrcr} x_1 & +x_2 & & +x_4 & = & 2 \\ & -x_2 & -x_3 & +x_4 & = & -1 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 0. \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$
$\begin{array}{rrrrcr} -x_1 & +2x_2 & & & = & -2 \\ x_1 & +x_2 & & +x_4 & = & 2 \\ & -x_2 & -x_3 & +x_4 & = & -1 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 0. \end{array}$	$\begin{pmatrix} 9/2 \\ 5/4 \\ -4 \\ -15/4 \end{pmatrix}$

4 The Water Supply Problem

Consider the water supply system of a given city, represented in Figure 1, that consists of the following elements (see Figure 1):

Pipes: They represent the paths to be followed by the water.

Nodes: They are the points where the pipes intersect, and where water enters or leaves the flow network. We assume that there are two supply nodes, those coinciding with the two deposits, and the remaining nodes are consumption nodes.

Data: The data of our problem are the amounts of flow that enter or leave each node, indicated by the arrows, and the topology of the network, depicted in Figure 1.

Unknowns: The unknowns in the water supply problem are the water flows in each of the pipes, that is, the number of unknowns coincides with the number of pipes.

Equations: To derive the system of equations that model this problem we must establish the fluid balance at each node, i.e., assuming that there are no losses, the income flow + the output flow must be null at each node. This reveals that we have a system of equations with as many equations as nodes.

So, the diagram in Figure 1 can be completed by numbering nodes and pipes (unknowns), and associating to each node i its input or output flow q_i, to get the diagram in Figure 2. Note that an arbitrary direction has been assigned to each flow. If the resulting variable is positive this means that the direction of the flow coincides with the arbitrary assignment; otherwise, it is contrary to it.

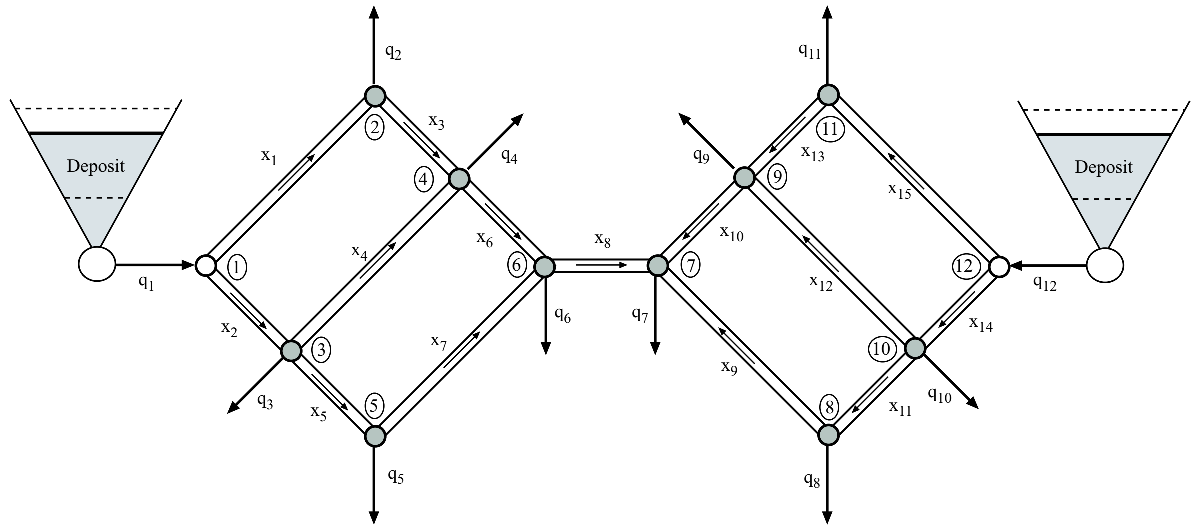

Figure 2: The water supply system of a given city showing the unknowns, i.e., the flows associated with all pipes, and the node and flow numberings.

Establishing the node balance equations, in matrix form, we get:

$$\begin{bmatrix} -1 & -1 & & & & & & & & & & & & & \\ 1 & & -1 & & & & & & & & & & & & \\ & 1 & & -1 & -1 & & & & & & & & & & \\ & & 1 & 1 & & -1 & & & & & & & & & \\ & & & & 1 & & -1 & & & & & & & & \\ & & & & & 1 & 1 & -1 & & & & & & & \\ & & & & & & & 1 & 1 & 1 & & & & & \\ & & & & & & & & -1 & & 1 & & & & \\ & & & & & & & & & -1 & & 1 & 1 & & \\ & & & & & & & & & & -1 & -1 & & 1 & \\ & & & & & & & & & & & & -1 & & 1 \\ & & & & & & & & & & & & & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix} = \begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{bmatrix} \quad (13)$$

that is a nice looking and convenient banded matrix, which columns contain a one and a minus one, because each pipe has associated flow entering one node and leaving another node.

It is important to realize about the importance of the node numbering, that has a great influence on the structure of the coefficients matrix. For example, assume that we number the nodes as in Figure 3,

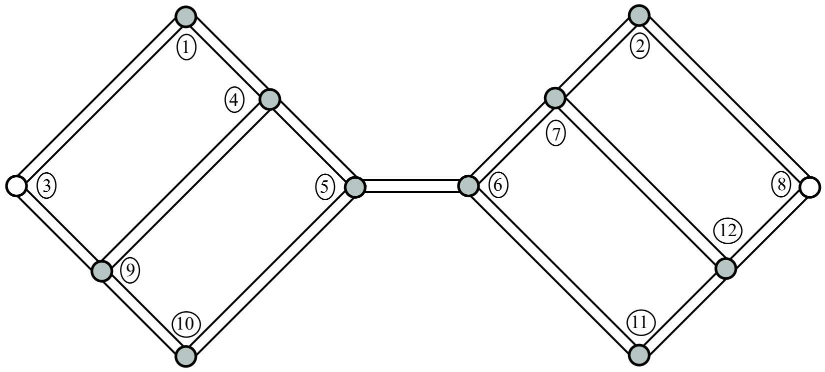

Figure 3: Alternative numbering of the nodes (not appropriate).

then, the resulting system of equations does not lead to a banded matrix any more and becomes:

$$
\begin{bmatrix}
1 & & -1 & & & & & & & & & & & & \\
 & & & & & & & & & & & & -1 & & 1 \\
-1 & -1 & & & & & & & & & & & & & \\
 & & 1 & 1 & & -1 & & & & & & & & & \\
 & & & & & 1 & 1 & -1 & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & & & -1 & & 1 & 1 & & \\
 & & & & & & & & & & & & & -1 & -1 \\
 & 1 & & -1 & -1 & & & & & & & & & & \\
 & & & & 1 & & -1 & & & & & & & & \\
 & & & & & & & & -1 & & 1 & & & & \\
 & & & & & & & & & & -1 & -1 & & 1 &
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}
=
\begin{bmatrix} q_1 \\ q_2 \\ -q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ -q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{bmatrix}. \tag{14}
$$

Note that it is convenient that connected nodes have close numbers; otherwise the matrix band size increases. Thus, from now on we will use the numbering in Figure 2.

First, we can ask ourselves for the conditions to be satisfied for the linear system (13) to have solution (one or more). Then, we need to use the method described in Section 3.1, that leads to the conditions

$$-q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7 + q_8 + q_9 + q_{10} + q_{11} - q_{12} = 0.$$

Note that they express that the amount of water entering the network must coincide with the amount of water leaving the network. Thus, the engineering meaning of the compatibility condition is a global balance of flow.

However, note that this compatibility holds for this topology, but is not the general compatibility condition. For example, in Figure 4 we show the case in which pipes 6,7,9 and 10 have been removed, for which satisfaction of the global balance is not sufficient for compatibility. In fact, the new system of equations for this case is:

$$\begin{bmatrix} -1 & -1 & & & & & & & & & \\ 1 & & -1 & & & & & & & & \\ & 1 & & -1 & -1 & & & & & & \\ & & 1 & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & -1 & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & 1 & & \\ & & & & & & -1 & -1 & & 1 & \\ & & & & & & & & -1 & & 1 \\ & & & & & & & & & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_8 \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix} = \begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{bmatrix}$$

and the compatibility conditions become:

$$\begin{aligned} -q_1 + q_2 + q_3 + q_4 + q_5 &= 0 \\ q_6 + q_7 &= 0 \\ q_8 + q_9 + q_{10} + q_{11} - q_{12} &= 0 \end{aligned} \tag{15}$$

that represent the global balances for each of the three existing subnetworks (see Figure 4).

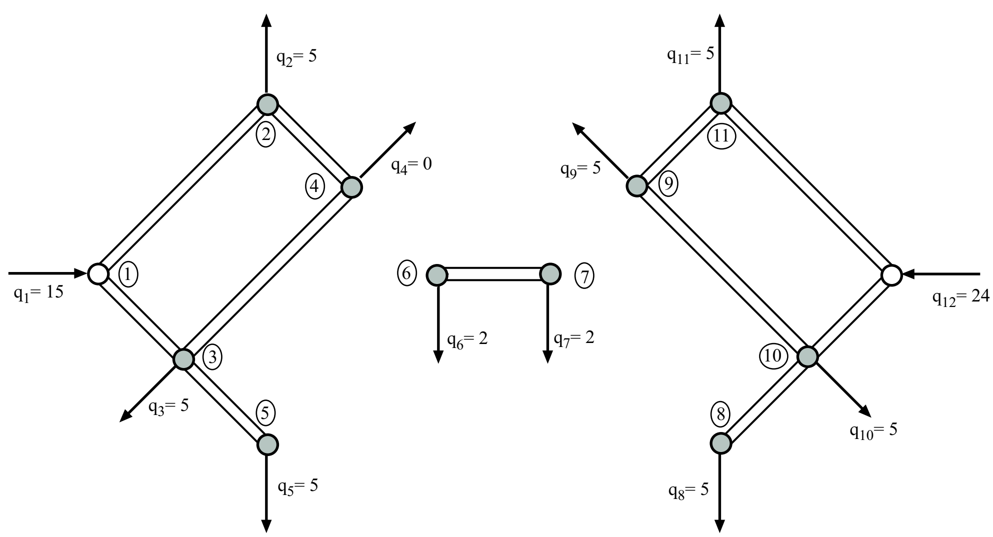

Figure 4: An example where the global balance is not sufficient for compatibility.

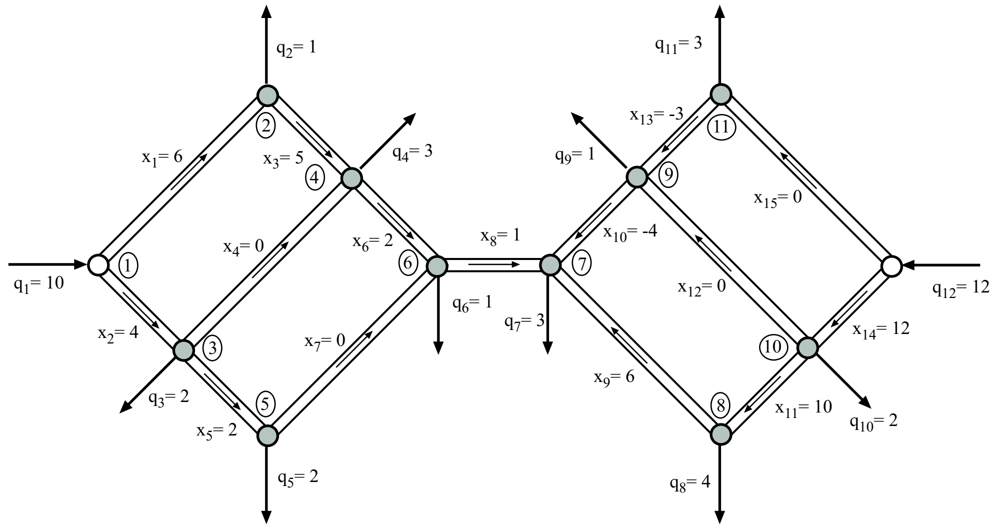

Figure 5: A particular numerical case where the deposits supply 10 and 12 units of flow and the node consumptions are indicated.

Consider the particular case in Figure 5. To obtain the set of all possible solutions, we can apply the orthogonalization algorithm 1 and obtain:

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9\\x_{10}\\x_{11}\\x_{12}\\x_{13}\\x_{14}\\x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\-1\\1\\-1\\1\\0\\0\\0 \end{pmatrix} + \rho_2 \begin{pmatrix} -1\\1\\-1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} + \rho_3 \begin{pmatrix} -1\\1\\-1\\0\\1\\-1\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} + \rho_4 \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\-1\\1\\-1\\0\\1\\-1\\1 \end{pmatrix} + \begin{pmatrix} 6\\4\\5\\0\\2\\2\\0\\1\\6\\-4\\10\\0\\-3\\12\\0 \end{pmatrix}, \tag{16}$$

where $\rho_1, \rho_1, \ldots, \rho_4 \in \mathbb{R}$, that from a mathematical point of view is an affine linear space, i.e., the sum of a given vector (the last one) plus a linear space of dimension 4 (an arbitrary linear combination of 4 linearly independent vectors).

From an engineering point of view, this solution must be interpreted as follows:

1. The given vector is a particular solution, i.e., a solution to the stated problem. Note that it satisfies Equation (13) for the q values in Figure 5. This vector can be replaced by any other particular solution.

2. The first vector corresponds to a solution of the associated homogeneous problem, i.e., assuming that no flows enters or leaves the network, and that the flow is null with the exception of pipes 9, 10, 11 and 12, that is represented in Figure 6(a).

3. The second vector corresponds to a solution of the associated homogeneous problem, i.e., assuming that no flows enters or leaves the network, and that the flow is null with the exception of pipes 1, 2, 3 and 4, that is represented in Figure 6(b).

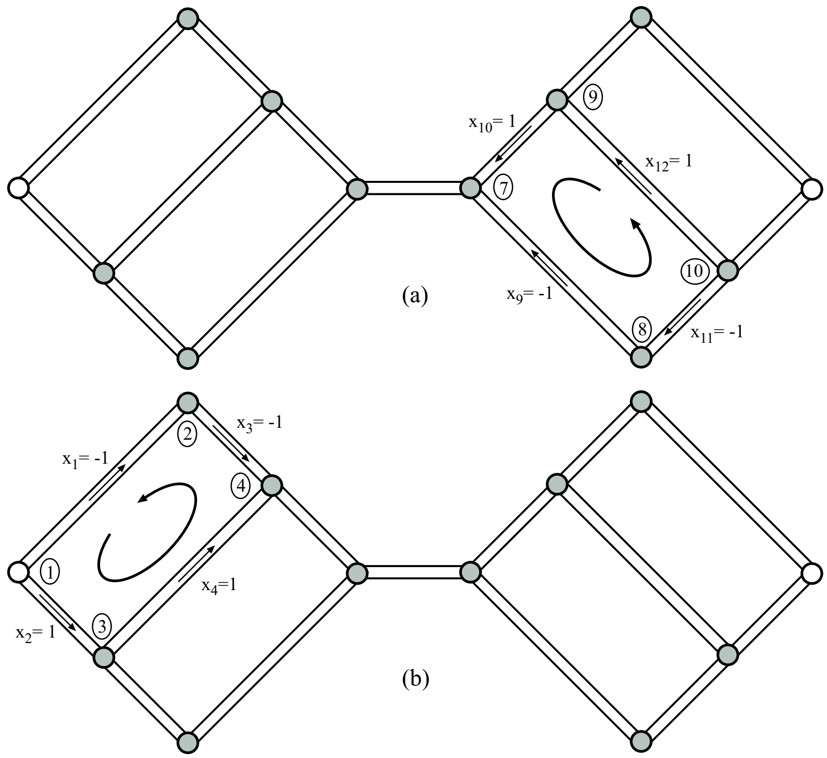

Figure 6: A particular solution of the associated homogeneous system of equations. Note that no flow enters or leave the network, and inner flow affects only to some pipes.

4. The third vector corresponds to a solution of the associated homogeneous problem, i.e., assuming that no flows enters or leaves the network, and that the flow is null with the exception of pipes 1, 2, 3, 5, 6 and 7, that is represented in Figure 7(a).

5. The fourth vector corresponds to a solution of the associated homogeneous problem, i.e., assuming that no flows enters or leaves the network, and that the flow is null with the exception of pipes 9, 10, 11, 13, 14 and 15, that is represented in Figure 7(b).

The linear space generated by the first four vectors in (16) can be represented using another basis of the same space; in particular the basic vectors represented in Figure 6 or those in Figure 7 can be replaced by those represented in Figure 8.

The appearance of ρ values in the general solution (16) implies that the flows in the pipes can be unlimited in any direction (the ρ's can be positive or negative). This has no physical sense and will be corrected in Section 7. The number of ρ's, i.e., the dimension of the linear space of homogeneous solutions is related to the number of degrees of freedom of our solution, that is, the maximum number of redundant pipes, i.e., that can be removed or fail without failure in the required water supply.

To know whether or not a set of pipes can be removed, we just need to enforce the flow in all pipes in the set to be null. For example, assume that we want to know if pipes 1 and 2 can be removed, then, from (16) we must have:

$$
\begin{aligned}
x_1 &= -\rho_2 - \rho_3 + 6 = 0 \\
x_2 &= \rho_2 + \rho_3 + 4 = 0
\end{aligned}
$$

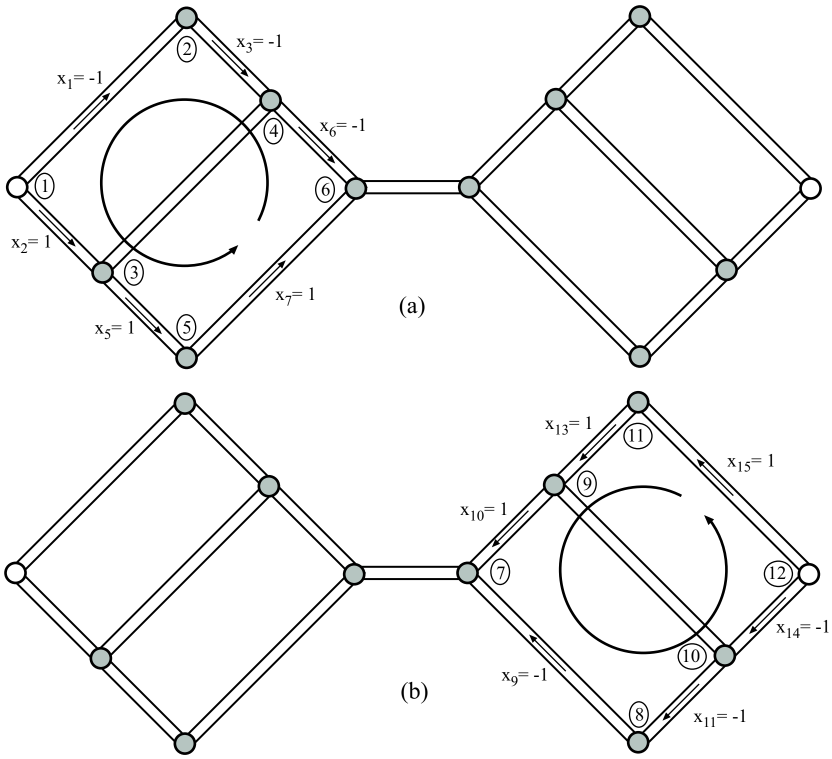

Figure 7: A particular solution of the associated homogeneous system of equations. Note that no input or output flow occurs, and inner flow affects only to some pipes.

that is an incompatible system of equations, because the submatrix

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

associated with x_1 and x_2 has rank 1.

Thus, the general rule to determine if a set of pipes can simultaneously fail without influencing the water supply is that the rank of the associated matrix must coincide with the number of pipes in that set. For example, the pipes $4, 7, 9$ and 12 lead to the matrix (see (16)):

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

that has rank 4 indicating that they can be removed simultaneously without affecting the required supply. The solution corresponds to $\rho_1 = \rho_2 = \rho_3 = 0$ and $\rho_4 = 6$, that leads to the particular solution:

$$(6, 4, 5, 0, 2, 2, 0, 1, 0, 2, 4, 0, 3, 6, 6)^T$$

5 The Γ algorithm

In this section we describe the Γ algorithm, (see Jubete [12, 13], Padberg [16], Castillo, Jubete, Pruneda and Solares [5], Castillo, Esquivel y Pruneda [6] and Castillo, Conejo, Pedregal, García and Alguacil [4]), that is the key algorithm to solve systems of inequalities.

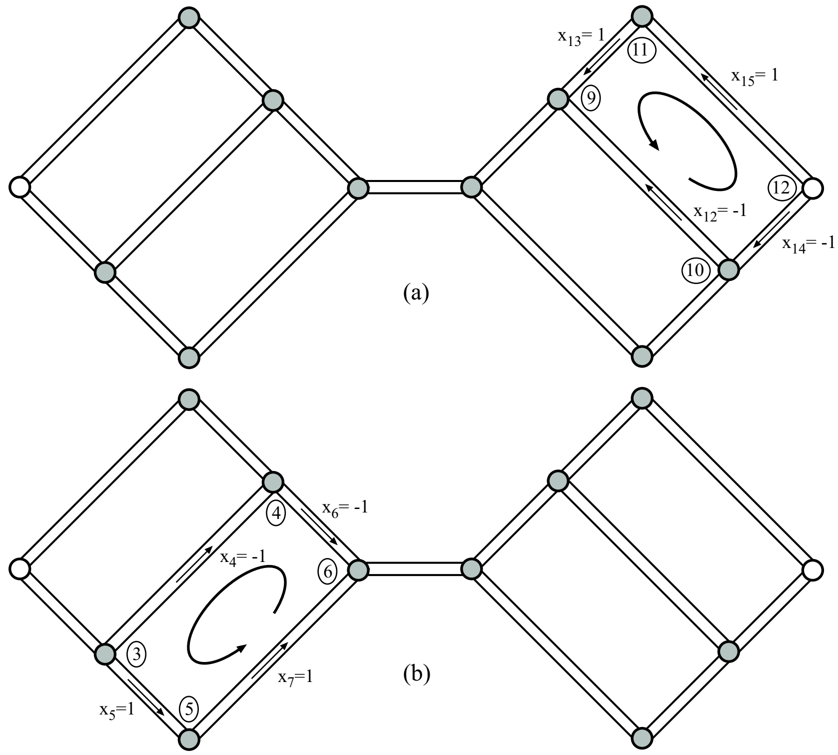

Figure 8: Alternative basis for the linear space of dimension 4 appearing in the general solution.

The reader interested in a classical treatment of some of these problems can, for example, consult the works of Minkowski [14]), Motzkin, Raiffa, Thompson, and Thrall [15], Chazelle [7], Chernikova [8] and Greenberg [11], Dyer [9], Pillers [17], etc.

Since the concepts of cone and dual cone are used, we start with their definitions.

Definition 1 (Polyhedral convex cone) *Let* $\mathbf{A}$ *be a matrix, and* $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ *be its column vectors. The set*

$$\mathbf{A}_\pi \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \pi_1 \mathbf{a}_1 + \ldots + \pi_m \mathbf{a}_m \quad \textit{with} \quad \pi_j \geq 0; j = 1, \ldots, m\}$$

of all nonnegative linear combinations of the column vectors of $\mathbf{A}$ *is known as the polyhedral convex cone generated by* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *(its generators), and is denoted* $\mathbf{A}_\pi$*.*

A cone $\mathbf{A}_\pi$ can be written as the sum of a linear space $\mathbf{V}_\rho$ plus a pure cone $\mathbf{W}_\pi$, i.e., $\mathbf{A}_\pi = \mathbf{V}_\rho + \mathbf{W}_\pi$. In this paper we use the Greek letter π to refer to non-negative real numbers.

Definition 2 (Nonpositive dual or polar cone) *Let* $\mathbf{A}_\pi$ *be a cone in* $\mathbb{R}^n$ *with generators* $\mathbf{a}_1, \ldots, \mathbf{a}_k$*. The nonpositive dual of* $\mathbf{A}_\pi$*, denoted* $\mathbf{A}_\pi^p$*, is defined as the set*

$$\mathbf{A}_\pi^p \equiv \left\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}^T \mathbf{u} \leq \mathbf{0}\right\} \equiv \left\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{a}_i^T \mathbf{u} \leq 0;\ i = 1, \ldots, k\right\}$$

that is, the set of all vectors such that their dot products by all vectors in $\mathbf{A}_\pi$ *are nonpositive.*

Algorithm 2 (Dual cone of a given cone.)

- **Input:** A cone defined by a non-necessarily minimal set of generators $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in $\mathbb{R}^n$ that is partitioned in two sets: $\mathbf{B}$ and $\mathbf{C}$, such that the cone be in standard form $\mathbf{A}_\pi = \mathbf{B}_\rho + \mathbf{C}_\pi$.
- **Output:** The dual cone in one of its minimal representations.

Initialization:

- Since at iteration h we look for a minimal set of generators of the dual cone generated by $\{\mathbf{a}_1, \ldots, \mathbf{a}_h\}$ in the form $\mathbf{V}_\rho^h + \mathbf{W}_\pi^h$, the matrix $\mathbf{U}^h$ of the generators of $\mathbf{A}_\pi^h$ at iteration h, will be partitioned as $(\mathbf{V}^h, \mathbf{W}^h)$, where the columns of $\mathbf{V}^h$ and $\mathbf{W}^h$ are the linear space and the acute cone generators of the corresponding dual cone, respectively.

  Initially, i.e., when no $\mathbf{a}$ vectors have been considered yet, the dual cone is $\mathbb{R}^n = (\mathbf{I_n})_\rho$, where $\mathbf{I}_n$ is the identity matrix of dimension n. Then, we let $\mathbf{V}^1 = \mathbf{I}_n$, $\mathbf{W}^1 = \emptyset$, and $\mathbf{U}^1 = (\mathbf{V}^1, \mathbf{W}^1)$.

- Since for minimal representation purposes, each vector $\mathbf{u}_j^h$ in $\mathbf{U}^h$ will be assigned at the end of iteration h the set
$$I_j^h = \{1 \le i \le h | \mathbf{a}_h^T \mathbf{u}_j^h = 0\},$$
  we initialize the set I_j^1 to empty sets, for $j = 1, 2, \ldots, n$, and let the iteration number $h = 1$ (first iteration).

Regular Process:

Step 1: Calculate the dot products. Calculate $\mathbf{t}^h = \mathbf{a}_h^T \mathbf{U}^h$.

Step 2: Look for the pivot. Find a column $\mathbf{v}_i$ (called a pivot column) in $\mathbf{V}^h$ such that $t_i^h \neq 0$.

Step 3: Test for Γ_I or Γ_{II} processes. If no pivot has been found, go to **Process II** (Step 5). Otherwise go to **Process I** (Step 4).

Step 4: Process I. Normalize the pivot column by dividing it by $-t^h_{\text{pivot}}$. Perform the pivoting process by letting $u_{ij}^h = u_{ij}^h + t_j^h u_i^h$ for all $j \neq$ pivot. Append the index h to the I_j^h sets for all $j \neq$ pivot. If $\mathbf{a}_h \in \mathbf{B}$, remove vector $\mathbf{u}_{pivot}^h$ from $\mathbf{V}^h$ and go to Step 6. Otherwise, remove $\mathbf{u}_{pivot}^h$ from $\mathbf{V}^h$, and append it to $\mathbf{W}^h$. Then, go to Step 6.

Step 5: Process II. Append to I_j^h the index h for all j such that $t_j = 0$.

For all $\mathbf{w}_j \in \mathbf{W}^h$ divide $\mathbf{w}_j$ by $|t_j^h|$.

Consider the set of vectors
$$Z = \{\mathbf{w}_{k(i,j)} = \mathbf{w}_i + \mathbf{w}_j | t_i^h < 0, t_j^h > 0; \mathbf{w}_i, \mathbf{w}_j \in \mathbf{W}^h\}.$$

Assign the vectors in Z the sets $I_{k(i,j)}^h = (I_i^h \cap I_j^h) \cup \{h\}$, and select from Z a maximal subset $Z^* \subset Z$ of vectors $\mathbf{w}_{k(i,j)}$ such that $I_{k(i,j)}^h \not\subset I_{k(i_1,j_1)}^h$ and $I_{k(i,j)}^h \not\subset I_s^h$, for all $\mathbf{w}_s$ such that $t_s^h = 0$.

Remove from $\mathbf{W}^h$ all $\mathbf{w}_j$ such that $t_j^h > 0$ and append to $\mathbf{W}^h$ all vectors of Z^*.

Step 6: If $h < m$, let $h = h + 1$ and go to Step 1; otherwise, return matrices $\mathbf{V}^h$ and $\mathbf{W}^h$, and exit.

Remark 3 *At the end of each iteration of the Γ-algorithm,i.e., after Step 5, the* $\mathbf{v}$ *column vectors of the tableau can be multiplied by any non-negative number, and the* $\mathbf{W}$ *column vectors of the tableau can be multiplied by any positive number.*

6 Solving Systems of Inequalities

6.1 Deciding whether or not a system of linear inequalities is compatible

In this section we show how to apply the Γ-algorithm to analyze the compatibility of a system of linear inequalities.

Consider the system:
$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & b_1, \\ a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & b_2, \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \\ a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & b_m \\ & & & & x_1, x_2, \cdots, x_n & \geq & 0 \end{array} \tag{17}$$

that can be written as

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad (18)$$
$$x_1, x_2, \cdots, x_n \geq 0.$$

Expression (18) shows that the given system is compatible if and only if the vector $\mathbf{b} = (b_1, \ldots, b_m)^T$ belongs to the cone generated by the set of column vectors $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ of the coefficient matrix $\mathbf{A}$, i.e.,

$$\mathbf{b} \in \mathbf{A}_\pi \equiv \mathbf{b} \in (\mathbf{A}_\pi^p)^p \,. \qquad (19)$$

Thus, the compatibility problem reduces to finding the dual cone $\mathbf{V}_\rho + \mathbf{W}_\pi$ of the cone generated by the columns of the coefficient matrix and checking that $\mathbf{b}^T\mathbf{V} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{W} \leq \mathbf{0}$.

To analyze the compatibility of a system of linear inequalities in arbitrary variables, it can be converted to the case in (17), using slack variables to convert the inequalities in equalities, and one more artificial variable to convert the arbitrary variables into no negative variables, that is, each variable x_i can be converted to $x_i^* - x_0$.

Example 4 (Compatibility of a linear system of equations in restricted variables) *Determine the conditions for the following system of equations to be compatible.*

$$\begin{array}{rrrrrrcl} & -x_2 & & +2x_4 & +x_5 & -x_6 & = & a \\ & & x_3 & +x_4 & & +2x_6 & = & b \\ & x_2 & -x_3 & & +x_5 & +2x_6 & = & c \\ -x_1 & +x_2 & -x_3 & & -x_5 & -x_6 & = & d \\ & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0. \end{array} \qquad (20)$$

For the system (20) to be compatible, the vector $(a, b, c, d)^T$ must belong to the cone $\mathbf{A}_\pi$ generated by the columns of the coefficient matrix, that is to say, it must belong to the dual of the dual of $\mathbf{A}_\pi$. In other words, their dot products by the cone generators of $\mathbf{A}_\pi^p$ must be non-positive. Since $\mathbf{A}_\pi^p$ can be obtained from the Table 5, we obtain the desired compatibility condition:

$$\begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \leq \mathbf{0}. \qquad (21)$$

6.2 Solving a homogeneous system of linear inequalities

Consider the homogeneous system of inequalities

$$\begin{array}{llllcl} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & \leq & 0, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & \leq & 0, \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \leq & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & \leq & 0 \end{array} \qquad (22)$$

which can be written as

$$\begin{array}{rcl} (a_{11}, \ldots, a_{1n})(x_1, \ldots, x_n)^T & \leq & 0, \\ (a_{21}, \ldots, a_{2n})(x_1, \ldots, x_n)^T & \leq & 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \leq & \cdots \\ (a_{m1}, \ldots, a_{mn})(x_1, \ldots, x_n)^T & \leq & 0. \end{array} \qquad (23)$$

Expression (23) shows that $(x_1, \ldots, x_n)$ is the dual cone of the row vectors $\{\mathbf{a}^1, \mathbf{a}^1, \ldots, \mathbf{a}^m\}$ of $\mathbf{A}$.

Thus, obtaining the solution of the system (22) reduces to determining the dual cone $\mathbf{A}_\pi^p$ of the cone generated by the cone generated by the rows of matrix $\mathbf{A}$.

Iteration 1				
$\mathbf{a}_1$	$\mathbf{v}_1^1$	$\mathbf{v}_2^1$	$\mathbf{v}_3^1$	$\mathbf{v}_4^1$
0	1	0	0	**0**
0	0	1	0	**0**
0	0	0	1	**0**
-1	0	0	0	**1**
$\mathbf{t}^1$	0	0	0	**-1**
I_j^1	1	1	1	

Iteration 2				
$\mathbf{a}_2$	$\mathbf{v}_1^2$	$\mathbf{v}_2^2$	$\mathbf{v}_3^2$	$\mathbf{w}_1^2$
-1	1	0	**0**	0
0	0	1	**0**	0
1	0	0	**1**	0
1	0	0	**0**	1
$\mathbf{t}^2$	-1	0	**1**	1
I_j^2	1	1	1	
		2		

Iteration 3				
$\mathbf{a}_3$	$\mathbf{v}_1^3$	$\mathbf{v}_2^3$	$\mathbf{w}_1^3$	$\mathbf{w}_2^3$
0	1	**0**	0	0
1	0	**1**	0	0
-1	1	**0**	-1	-1
-1	0	**0**	0	1
$\mathbf{t}^3$	-1	**1**	1	0
I_j^3	1	1	1	2
	2	2		3

Iteration 4				
$\mathbf{a}_4$	$\mathbf{w}_1^4$	$\mathbf{v}_1^4$	$\mathbf{w}_2^4$	$\mathbf{w}_3^4$
2	**1**	0	0	0
1	**1**	-1	-1	0
0	**1**	0	-1	-1
0	**0**	0	0	1
$\mathbf{t}^4$	**3**	-1	-1	0
I_j^4	1	1	1	2
	2	2	3	3
	3			4

Iteration 5				
$\mathbf{a}_5$	$\mathbf{w}_1^5$	$\mathbf{w}_2^5$	$\mathbf{w}_3^5$	$\mathbf{w}_4^5$
1	-1/3	1/3	1/3	0
0	-1/3	-2/3	-2/3	0
1	-1/3	1/3	-2/3	-1
-1	0	0	0	1
$\mathbf{t}^5$	-2/3	2/3	-1/3	-2
I_j^5	1	1	1	2
	2	2	3	3
	3	4	4	4

Iteration 6						
$\mathbf{a}_6$	$\mathbf{w}_1^6$	$\mathbf{w}_2^6$	$\mathbf{w}_3^6$	$\mathbf{w}_4^6$	$\mathbf{w}_5^6$	$\mathbf{w}_6^6$
-1	-1/3	1/3	0	2/3	1/3	0
2	-1/3	-2/3	0	-4/3	-2/3	-2/3
2	-1/3	-2/3	-1	0	-1/3	0
-1	0	0	1	2/3	0	0
$\mathbf{t}^6$	-1	-3	-3	-4	-7/3	-4/3
I_j^6	1	1	2	2	1	1
	2	3	3	4	4	2
	3	4	4	5	5	5

Normalized dual						
	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}_5$	$\mathbf{w}_6$
	-1	1	0	1	1	0
	-1	-2	0	-2	-2	-1
	-1	-2	-1	0	-1	0
	0	0	1	1	0	0

Table 5: Gamma Process to determine the dual cone in Example 4.

Example 5 (Solving an homogeneous system of linear inequalities) *Consider the system of equations*

$$\begin{array}{rrrrl} & & & -x_4 & \leq 0 \\ -x_1 & & +x_3 & +x_4 & \leq 0 \\ & x_2 & -x_3 & -x_4 & \leq 0 \\ 2x_1 & +x_2 & & & \leq 0 \\ x_1 & & +x_3 & -x_4 & \leq 0 \\ -x_1 & +2x_2 & +2x_3 & -x_4 & \leq 0 \end{array} \tag{24}$$

To solve this system, we obtain the dual cone of the cone generated by the rows coefficients, as shown in Table 5. Thus, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix},$$

where π_1 to π_6 are arbitrary non-negative real numbers. From Table 5 other subsets of inequalities in (24) can be solved, as shown in Table 2.

Table 6: Some examples of homogeneous subsystems of (24) that can be solved from Table 5.

System	Solution
$-x_4 \le 0$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} + \pi_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
$\begin{array}{rrrrl} & & & -x_4 & \le 0 \\ -x_1 & & +x_3 & +x_4 & \le 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$
$\begin{array}{rrrrl} & & & -x_4 & \le 0 \\ -x_1 & & +x_3 & +x_4 & \le 0 \\ & x_2 & -x_3 & -x_4 & \le 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$
$\begin{array}{rrrrl} & & & -x_4 & \le 0 \\ -x_1 & & +x_3 & +x_4 & \le 0 \\ & x_2 & -x_3 & -x_4 & \le 0 \\ 2x_1 & +x_2 & & & \le 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -2 & -2 & 0 \\ -1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix}$
$\begin{array}{rrrrl} & & & -x_4 & \le 0 \\ -x_1 & & +x_3 & +x_4 & \le 0 \\ & x_2 & -x_3 & -x_4 & \le 0 \\ 2x_1 & +x_2 & & & \le 0 \\ x_1 & & +x_3 & -x_4 & \le 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix}$
$\begin{array}{rrrrl} & & & -x_4 & \le 0 \\ -x_1 & & +x_3 & +x_4 & \le 0 \\ & x_2 & -x_3 & -x_4 & \le 0 \\ 2x_1 & +x_2 & & & \le 0 \\ x_1 & & +x_3 & -x_4 & \le 0 \\ -x_1 & +2x_2 & +2x_3 & -x_4 & \le 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix}$

6.3 Solving a complete system of linear inequalities

Now consider the complete system of linear inequalities:

$$\begin{array}{rrrrcl} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & \le & b_1, \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & \le & b_2, \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & \le & b_m. \end{array} \tag{25}$$

Adding the artificial variable x_{n+1}, the constraint $x_{n+1} = 1$ and the redundant constraint $x_{n+1} \ge 0$ (it is a key trick that allows the constraint $x_{n+1} = 1$ to be easily forced at the end of the process), it can be written as

$$\begin{array}{rrrrrcl} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & -b_1x_{n+1} & \le & 0 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & -b_2x_{n+1} & \le & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & -b_mx_{n+1} & \le & 0 \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & -b_mx_{n+1} & \le & 0 \\ & & & & -x_{n+1} & \le & 0 \\ & & & & x_{n+1} & = & 1. \end{array} \tag{26}$$

System (26) can be written as

$$\begin{array}{rcl} (a_{11},\ldots,a_{1n},-b_1)(x_1,\ldots,x_n,x_{n+1})^T & \leq & 0 \\ (a_{21},\ldots,a_{2n},-b_2)(x_1,\ldots,x_n,x_{n+1})^T & \leq & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots \\ (a_{m1},\ldots,a_{mn},-b_m)(x_1,\ldots,x_n,x_{n+1})^T & \leq & 0 \\ -x_{n+1} & \leq & 0 \\ x_{n+1} & = & 1. \end{array} \tag{27}$$

Expression (27) shows that $(x_1,\ldots,x_n,x_{n+1})$ belongs to the dual cone of the cone generated by the set of vectors

$$\{(a_{11},\ldots,a_{1n},-b_1),(a_{21},\ldots,a_{2n},-b_2),\ldots,(a_{m1},\ldots,a_{mn},-b_m),(0,0,\cdots,0,-1)\}.$$

Then, it is clear that the solution of (26) is the intersection of that cone with the hyperplane $x_{n+1}=1$. Thus, the solution of (25) is the projection on $X_1\times\cdots\times X_n$ of the solution of (26).

Example 6 (Solving a complete system of linear inequalities) *To solve the following system of inequalities:*

$$\begin{array}{rrrcr} -x_1 & & +x_3 & \leq & -1 \\ & x_2 & -x_3 & \leq & 1 \\ 2x_1 & +x_2 & & \leq & 0 \\ x_1 & & +x_3 & \leq & 1 \\ -x_1 & +2x_2 & +2x_3 & \leq & 1 \end{array} \tag{28}$$

we use the auxiliary variable x_4 and the redundant constraint $1=x_4\geq 0$. Then, the system (28) can be written as:

$$\begin{array}{rrrrl} & & & -x_4 & \leq 0 \\ -x_1 & & +x_3 & +x_4 & \leq 0 \\ & x_2 & -x_3 & -x_4 & \leq 0 \\ 2x_1 & +x_2 & & & \leq 0 \\ x_1 & & +x_3 & -x_4 & \leq 0 \\ -x_1 & +2x_2 & +2x_3 & -x_4 & \leq 0 \\ & & & x_4 & = 1. \end{array} \tag{29}$$

Since the upper part is an homogeneous system, one need to find the dual cone of the cone generated by the row coefficients, that appears in Table 5. After imposing condition $x_4=1$ one get the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{30}$$

Table 7 gives the solution of several subsystems from (28), that can be obtained from Table 5.

7 The Water Supply Problem Revisited

In this section we analyze the water supply problem, but assuming that there are some constraints to the flow.

First, we assume that there is a capacity limit in the pipes. Establishing the node balance equations,

Table 7: Some examples of complete subsystems of (28) that can be solved from Table 5.

System	Solution
$\begin{array}{rrcr} -x_1 & +x_3 & \le & -1 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \pi_1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
$\begin{array}{rrrcr} -x_1 & & +x_3 & \le & -1 \\ & x_2 & -x_3 & \le & 1 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \rho_1 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
$\begin{array}{rrrcr} -x_1 & & +x_3 & \le & -1 \\ & x_2 & -x_3 & \le & 1 \\ 2x_1 & +x_2 & & \le & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -2 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
$\begin{array}{rrrcr} -x_1 & & +x_3 & \le & -1 \\ & x_2 & -x_3 & \le & 1 \\ 2x_1 & +x_2 & & \le & 0 \\ x_1 & & +x_3 & \le & 1 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$\begin{array}{rrrcr} -x_1 & & +x_3 & \le & -1 \\ & x_2 & -x_3 & \le & 1 \\ 2x_1 & +x_2 & & \le & 0 \\ x_1 & & +x_3 & \le & 1 \\ -x_1 & +2x_2 & +2x_3 & \le & 1 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -2 & -2 & -1 \\ -1 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

in matrix form, we get:

$$\begin{bmatrix}
-1 & -1 & & & & & & & & & & & & & \\
1 & & -1 & & & & & & & & & & & & \\
 & 1 & & -1 & -1 & & & & & & & & & & \\
 & & 1 & 1 & & -1 & & & & & & & & & \\
 & & & & 1 & & -1 & & & & & & & & \\
 & & & & & 1 & 1 & -1 & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & & -1 & & 1 & & & & \\
 & & & & & & & & & -1 & & 1 & 1 & & \\
 & & & & & & & & & & -1 & -1 & & 1 & \\
 & & & & & & & & & & & & -1 & & 1 \\
 & & & & & & & & & & & & & -1 & -1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}
=
\begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{bmatrix} \tag{31}$$

subject to

$$-c_i \le x_i \le c_i; \quad i = 1, 2, \dots, 15, \tag{32}$$

where $c_i; i = 1, 2, \dots, 15$ are the pipe capacities.

In addition to $c_i \ge 0; i = 1, 2, \dots, 15$ the compatibility conditions of this system of inequalities, that have been obtained using the method described in Section 6.1, are:

$$-q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7 + q_8 + q_9 + q_{10} + q_{11} - q_{12} = 0 \tag{33}$$

and

$$
(-1)^p \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{pmatrix}
\leq
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \end{pmatrix}
\tag{34}
$$

for $p = 1, 2.$, which can be easily interpreted from an engineering point of view. The first condition is the global balance, and the remaining conditions are constraints associated with some partitions of the network. For example the constraints (see Figure 9 and the first boldfaced line in (34)):

$$
-c_1 - c_3 \leq q_2 \leq c_1 + c_3 \tag{35}
$$

establish that the total amount of flow entering the network cannot exceed the amount associated with the full capacity of the intersected pipes 1 and 3. Similarly, the constraints (see Figure 9 and the second boldfaced line in (34)):

$$
-c_2 - c_4 - c_7 \leq q_3 + q_5 \leq c_2 + c_4 + c_7 \tag{36}
$$

establish that the total amount of flow entering the network cannot exceed the amount associated with the full capacity of the intersected pipes 2, 4 and 7.

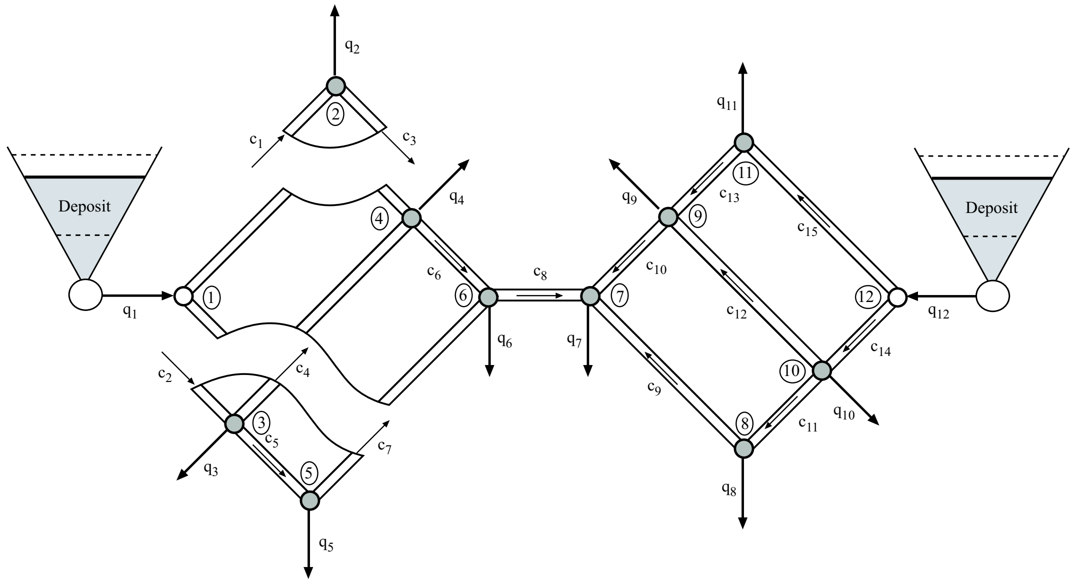

Figure 9: Illustration of the network partitions associated with the compatibility conditions $-c_1 - c_3 \leq q_2 \leq c_1 + c_3$ and $-c_2 - c_4 - c_7 \leq q_3 + q_5 \leq c_2 + c_4 + c_7$.

The solution of the system (31)-(32) with the flow data in Figure 5, using the algorithm for dual cones is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}; \quad \sum_{i=1}^{8} \lambda_i = 1; \quad \lambda_i \geq 0; i = 1, 2, \ldots, 8 \tag{37}$$

From this general solution many questions can be answered, as:

1. Which pipes are over dimensioned. They correspond to the pipes which components never reach its capacity in absolute value. Since in our example the capacity for all pipes has been assumed 6 flow units, we find that the over dimensioned pipes are: $x_3, x_7, x_8, x_9, x_{12}$ and x_{13} (see Figure 10). Note that the minimum required capacities for these pipes can also be obtained from the matrix in Figure 10.

2. Which pipes are critical (cannot fail). They correspond to pipes with components of the same sign in all the basic vectors. In our example they are the pipes x_8, x_{13}, x_{14} and x_{15} (see Figure 11).

3. Which pairs of pipes cannot fail simultaneously. For example, pipes associated with x_{10} and x_{11} cannot fail simultaneously (see Figure 12), because this would imply all λs to be null.

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix}
=
\begin{pmatrix}
4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\
6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\
3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\
6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\
-2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\
6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\
-4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\
6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\
4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{pmatrix}
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}
$$

Figure 10: Illustration of the process of locating the over dimensioned pipes.

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix}
=
\begin{pmatrix}
4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\
6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\
3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\
6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\
-2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\
6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\
-4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\
6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\
4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{pmatrix}
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}
$$

Figure 11: Illustration of the process of locating the critical pipes.

4. Which pipes have fixed flow. They have the same component in all basic vectors. In our example they correspond to pipes associated with x_8, x_{13}, x_{14} and x_{15} (see Figure 11).

It is also interesting to know the set of all possible solutions when some pipes fail. For example, if the

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \left(\begin{array}{cccccccc} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ \hline 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ \hline 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{array}\right) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}$$

Figure 12: Illustration of the process of locating pairs of pipes that cannot fail simultaneously.

pipe 10 fails to work, its flow will be null, and then the first four λs must be null, so that we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 \\ -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}; \quad \sum_{i=1}^{4} \lambda_i = 1; \quad \lambda_i \geq 0; i = 1, 2, \ldots, 4. \tag{38}$$

If pipes 7 and 10 do not work, we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \\ 5 & 3 \\ 0 & 2 \\ 2 & 2 \\ 2 & 2 \\ 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 6 & 6 \\ -2 & -2 \\ 3 & 3 \\ 6 & 6 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; \quad \sum_{i=1}^{2} \lambda_i = 1; \quad \lambda_i \geq 0; i = 1, 2 \tag{39}$$

and finally, if pipes 4, 7 and 10 do not work, we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 6 \\ -2 \\ 3 \\ 6 \\ 6 \end{pmatrix} \tag{40}$$

i.e., a unique solution, so that no further pipes can fail.

Next, we limit the direction of the flow using some retention valves, that allow the flow in only one direction, in pipes 2, 15, 14 and 1, sequentially, i.e., first a valve is used in pipe 2, then another valve is added in pipe 15, and then valves in pipes 14 and 1, are used.

The general solutions for the four cases are:

Retention valve in pipe 2:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \pi_1 + \begin{pmatrix} 10 \\ 0 \\ 9 \\ -4 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix} \tag{41}$$

that is the sum of an affine space of dimension 3 and a cone generated by a single vector. Note that $x_2 \geq 0$ and that the last column vector in (41) corresponds to a zero flow in pipe 2.

Retention valves in pipes 2 and 15:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 9 \\ -4 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix} \tag{42}$$

that is the sum of an affine space of dimension 2 and a cone generated by two vectors. Note that $x_2, x_{15} \geq 0$ and that the last column vector in (42) corresponds to a zero flow in pipes 2 and 15.

Retention valves in pipes 2, 15 and 14:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \pi_1 + \begin{pmatrix} 10 & 10 \\ 0 & 0 \\ 9 & 9 \\ -4 & -4 \\ 2 & 2 \\ 2 & 2 \\ 0 & 0 \\ 1 & 1 \\ 6 & -6 \\ -4 & 8 \\ 10 & -2 \\ 0 & 0 \\ -3 & 9 \\ 12 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{43}$$

that is the sum of a linear space of dimension 2, a cone generated by a single vector and a polytope with two vertices. Note that $x_2, x_{15}, x_{14} \geq 0$ and that the last two column vectors in (43) correspond to cases of zero and positive flows in pipes 2, 15 and 14.

Retention valves in pipes 2, 15, 14 and 1:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 \\ 9 & -1 & 9 & -1 \\ -4 & 6 & -4 & 6 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -6 & -6 & 6 & 6 \\ 8 & 8 & -4 & -4 \\ -2 & -2 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 9 & 9 & -3 & -3 \\ 0 & 0 & 12 & 12 \\ 12 & 12 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \qquad (44)$$

that is the sum of a linear space of dimension 2 and a polytope with four vertices. Note that $x_2, x_{15}, x_{14}, x_1 \geq 0$ and that the last two column vectors in (44) correspond to cases of zero and positive flows in pipes 2, 15, 14 and 1.

8 Conclusions

The following conclusions can be derived from this paper:

1. A full understanding of real problems stated as systems of linear equations or inequalities requires both the mathematical and the engineering points of view that complement each other.

2. The compatibility conditions must be interpreted from an engineering point of view, that help to identify errors, omissions or possible discrepancies between the mathematical model and the reality being modelled.

3. The mathematical structures of the general solutions, linear spaces, cones, polytopes and mixed combinations of these three structures have clear engineering interpretations that are closely related to the real problem being modelled.

4. The generators of the solution set, i.e., the linear space generators (basis), the cone generators, and the polytope generators (vertices) have clear interpretations from an engineering point of view, and contains a valuable information on the general solution of the problem and its properties.

Acknowledgments

The authors are indebted to the Spanish Ministry of Science and Technology (Project DPI2002-04172-C04-02) for partial support, and to the Editor, Associate Editor, and referees for the so many good and constructive comments and suggestion which have lead to a substantially improvement manuscript.

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