Design and Sensitivity Analysis Using the Probability-Safety-Factor Method. An Application to Retaining Walls

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Abstract

This paper presents a new method for designing engineering works that makes the classical approach, based on safety factors, and the modern, probability-based, approach compatible, and includes a sensitivity analysis. The method consists of a sequence of classical designs, based on given safety factors, that (a) minimizes cost or optimizes an alternative objective function, (b) calculates the different failure mode probabilities or their upper bounds, and (c) updates the safety factors to satisfy both the safety factors and the failure probability requirements. The process is repeated until convergence. As a result, an automatic design of the engineering work, the safety factors and the corresponding probabilities of failure for all failure modes are obtained. A double safety check is used and the correspondence between safety factors and probabilities of failure for the different modes are easily understood. An advantage of this approach is that the optimization procedure and the reliability calculations are decoupled. In addition, a sensitivity analysis is performed using a method that consists of transforming the data parameters into artificial variables and using the dual associated problem. The method is illustrated by its application to a retaining wall design.

Key Words: Level II and III, Optimization, Probability of failure, Reliability analysis, Safety factors, Sensitivity.

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This paper presents a new method for designing engineering works that makes the classical approach, based on safety factors, and the modern, probability-based, approach compatible, and includes a sensitivity analysis. The method consists of a sequence of classical designs, based on given safety factors, that (a) minimizes cost or optimizes an alternative objective function, (b) calculates the different failure mode probabilities or their upper bounds, and (c) updates the safety factors to satisfy both the safety factors and the failure probability requirements. The process is repeated until convergence. As a result, an automatic design of the engineering work, the safety factors and the corresponding probabilities of failure for all failure modes are obtained. A double safety check is used and the correspondence between safety factors and probabilities of failure for the different modes are easily understood. An advantage of this approach is that the optimization procedure and the reliability calculations are decoupled. In addition, a sensitivity analysis is performed using a method that consists of transforming the data parameters into artificial variables and using the dual associated problem. The method is illustrated by its application to a retaining wall design.

Key Words: Level II and III, Optimization, Probability of failure, Reliability analysis, Safety factors, Sensitivity.

1 Introduction

There are two principal ways of dealing with the design of engineering works:

1. *The classical approach.* This is based on safety factors, which are used to guarantee the required safety of the structures to be designed. The engineer, when faced with the problem of designing an engineering work, identifies all possible failure modes and chooses the design variable values for the corresponding engineering work to make it reasonably safe with respect to these modes. The greater the damage associated with the failure mode, the greater the level of safety required for this mode (see EUROCODE [11], ROM [24]).

   A classical design fixes the values of the safety factors and chooses the values of the design variables to satisfy these safety conditions. All the variables involved are assumed to be deterministic.

2. *The probability-based approach.* This works with probabilities of failure. Normally, a global probability of failure is used as the basic design criteria. However, working with failure probabilities is difficult because (a) it requires the definition of the joint probability of all variables involved, and (b) the evaluation of the failure probability is not an easy task. The problem becomes even more difficult if several failure modes are analyzed, because the failure region is the union of the different failure mode regions, and regions defined as unions are difficult to deal with because of their irregular and non-differentiable boundaries (see Melchers [21]). As an alternative design criteria, the probabilities of failure for the different modes can be used. Nevertheless, one may easily obtain an upper bound for the global failure probability by summing all the failure mode probabilities.

A probability-based design checks that the selected design leads to failure probabilities below given upper bounds. Some or all the variables involved are assumed to be random (see Rackwitz and Fiessler [22], Wirsching and Wu [30], Wu, Burnside and Cruse [31], and Ditlevsen and Madsen [10]).

Nowadays, both approaches are questioned: the classical approach because it does not give a clear idea of how far we are from failure, and the probability-based approach because it is very sensitive to the assumed joint distribution and tail assumptions.

Defenders of the classical and probabilistic approaches have serious difficulties in working together because they speak different languages. In this paper, we defend the coexistence of safety factors and failure probabilities and present a method that solves this problem, limiting both safety factors and probabilities of failure for the proposed designs. The method consists of a sequence of optimal (in the sense of minimizing the cost or an alternative objective function) classical designs, based on sets of safety factors bounds, that are adequately modified in each iteration to satisfy both the safety factors and the failure mode probability bound requirements.

Several authors have previously used optimization techniques to deal with engineering design, as for example, Lorenz [18], Kim and H. Adeli [16], Adeli [1], Bhatti [2], Sarma and Adeli [27, 28], and Ringertz [23], Royset, Der Kiureghian and Polak [26], etc. However, in this paper we deal with a new methodology that simultaneously considers safety factors and probabilities of failure. In addition, a sensitivity analysis procedure is presented. A sensitivity analysis adds quality to a design and supplies very important information on the work being designed from the view point of cost and reliability.

The paper is organized as follows. Section 2 introduces the problem of wall design and presents the particular example of a retaining wall used in the paper. Some background concepts are refreshed in Section 3. The proposed method is described in Section 4. A method for obtaining the failure probabilities and cost sensitivities to data is given in Section 5. The method is illustrated by its application to a wall design in Section 6. Section 7 is devoted to conclusions. Finally, an appendix explains some detailed analysis of the wall example.

2 Retaining Wall design

2.1 Introduction

Retaining structures are designed to hold back soil where an abrupt change in ground elevation occurs. The retained material or backfill exerts a push on the structure and thus tends to overturn or slide it, or both. The stem, heel and toe of such a wall act as cantilever beams, which need to be designed to withstand the soil pressures. Thus, a wall design implies:

1. Performing the stability check for the structure.
2. Computing the maximum and minimum soil pressures present under the toe and heel, comparing them with the allowable soil pressure provided as data.
3. Designing the reinforcing steel for the toe, heel and stem considering the corresponding bending and shear.

There are several well-known commercial computer programs, such as Correct Surcharge, DDRW-1, EPRES, FREW, GRETA, GWALL, HEAVE, Kzero2, LPRES, RETAIN, Retaining Wall Design, RetWall, ReWaRD, SHEET, Sheetpile-2, SHORING, SPUNT-A2, SPW 911, STAWALL, UNIBEAR, WALLAP, etc., that compute the soil bearing pressures under the base of a wall supporting any kind of backfill material with additional surcharge and concentrated external loads acting on the wall. In addition, they analyze the stability of the whole structure and perform the concrete design based on several design methods, in turn based on the working or the ultimate strength.

These programs normally offer the option of considering other external loads applied to the wall in addition to the backfill pressure. Though a surcharge can be defined as an equivalent height of backfill, they calculate earth pressures at key locations down the retaining wall, bending moments, shear forces, wall displacements, ground settlements, etc., and implement safety factors.

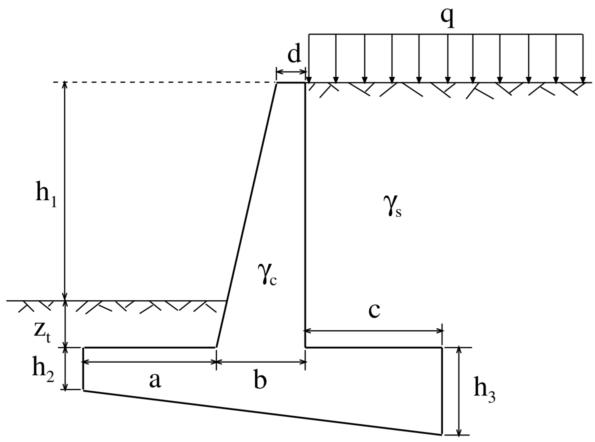

Figure 1: Geometrical description of the wall.

Some computer programs additionally design retaining walls according to certain international standards, such as BS 8002, Eurocode 7, CIRIA 104, Hong Kong Geoguide, CP2, British Steel's Piling Handbook and Highway Agency's BD42/94.

Most existing software for wall design uses the classical design based on safety factors, i.e. the design corresponds to level I. In other words, no random variables are allowed and, consequently, no probability of failure is obtained. Nowadays however the level II and III procedures have been substantially developed, and it is highly recommendable to use them.

An additional problem of these programs is that the design parameters, such as the toe or heel lengths, the stem widths, etc. have to be provided by the engineer, and an acceptable design of the wall is obtained only after a trial and error procedure. Once a wall design satisfying all the constraints is obtained, it is not easy to know whether we are far from or close to the optimal design. In this paper we deal with an optimal design that allows an automatic design of the wall.

2.2 Description of the retaining wall example

A retaining wall problem involves many variables, such as all wall dimensions, backfill slope, concrete and steel strength, allowable soil bearing pressure, backfill properties, etc. For example, consider the wall in Figure 1 that defines the geometry of the wall (note that the wall is defined in parametric form).

The main elements used in the wall design are:

1. **Geometric data.**

   h_1: height of the wall.

2. **Acting agents data.**

   q: surcharge.

3. **Material definition data.**

   γ_c: unit weight of the concrete.

   γ_s: unit weight of the soil.

   γ_{st}: unit weight of steel.

   f_c: concrete design strength.

   f_y: steel design strength.

   k_a: active pressure coefficient.

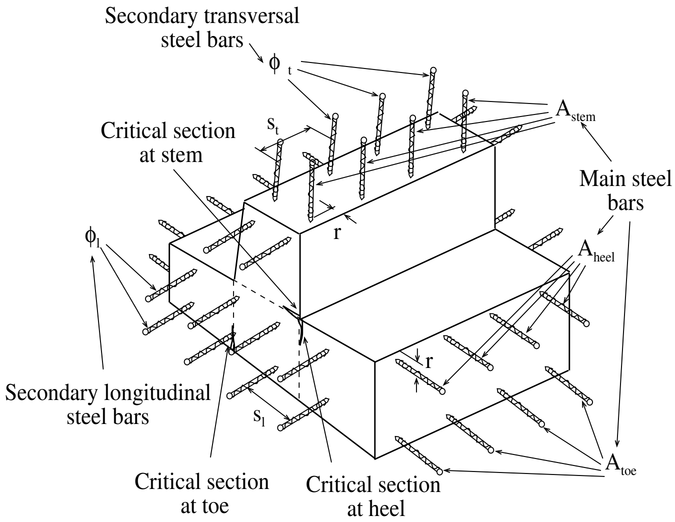

Figure 2: Illustration of the reinforced bar positions.

k_p: passive pressure coefficient.

σ_{soil}: soil strength.

μ_{crit}: critical friction factor.

τ_{max}: concrete shear strength.

4. **Steel bar data.** See Figure 2, which gives the details of the bar reinforcement.

r: bar cover depth.

ϕ_ℓ: secondary longitudinal steel bar diameter (minimum).

ϕ_t: secondary transversal steel bar diameter (minimum).

s_ℓ: secondary longitudinal bar spacing.

s_t: secondary transversal bar spacing.

5. **Mode safety factors lower bounds.**

F_t^0: safety factor against overturning.

F_s^0: safety factor against sliding.

F_b^0: safety factor for the bearing capacity constraint.

F_{stem}^0: safety factor for the bending moment at stem.

F_{toe}^0: safety factor for the bending moment at toe.

F_{heel}^0: safety factor for the bending moment at heel.

F_{sstem}^0: safety factor for the shear at stem.

F_{stoe}^0: safety factor for the shear at toe.

F_{sheel}^0: safety factor for the shear at heel.

6. **Cost data.**

c_c: cost per cubic meter of concrete.

c_{st}: cost per Newton of steel.

c_t: cost per square meter of timber.

c_{ex}: cost per cubic meter of excavation.

7. **Design variables.** These are the variables whose mean values are automatically selected by the optimization procedure.

   a: toe length.

   c: heel length.

   b: stem width, bottom.

   d: stem width, top.

   z_t: soil cover.

   h_2: toe height

   h_3: heel height.

   A_{stem}: cross section of the reinforcing bars at stem.

   A_{toe}: cross section of the reinforcing bars at toe.

   A_{heel}: cross section of the reinforcing bars at heel.

3 Classical and probability-based designs

In this section we introduce some basic concepts that are needed to understand the subsequent material.

The design and reliability analysis of an engineering work involves a number of random variables $(X_1, \ldots, X_n)$. These include geometric variables, material properties, loads, etc. In this paper we use uppercase letters to refer to random variables, and the corresponding lowercase letters to refer to particular instantiations of these variables. They belong to an n-dimensional space, which can be divided into two regions, the safe and failure regions:

$$\begin{array}{ll} \text{Safe Region:} & \mathcal{S} \equiv \{(x_1, x_2, \ldots, x_n)\} | g(x_1, x_2, \ldots, x_n) \geq 1\}, \\ \text{Failure Region:} & \mathcal{F} \equiv \{(x_1, x_2, \ldots, x_n)\} | g(x_1, x_2, \ldots, x_n) < 1\}. \end{array} \tag{1}$$

where $g(x_1, x_2, \ldots, x_n)$ can be the non-dimensional ratio of two opposing magnitudes, such as stabilizing to overturning forces, strengths to ultimate stresses, etc. Since the constraint $g(x_1, x_2, \ldots, x_n) = 1$ defines strict stability or security, to increase safety, the constant 1 is normally replaced by a larger constant F^0. If m different modes of failure are considered, the problem modifies to

$$\text{Design Region in mode i:} \quad \mathcal{S}_i \equiv \{(x_1, x_2, \ldots, x_n)\} | g_i(x_1, x_2, \ldots, x_n) \geq F_i^0\}, \tag{2}$$

where $i = 1, 2, \ldots, m$

It is important to distinguish between design values (those desired by the engineer), which in this paper are assumed to be the expectations, $E(X_i)$ or $\bar{x}_i$, or characteristic values, of the random variables $X_i : i = 1, 2, \cdots, n$, and actual values x_i (those existing in reality). Some of these expectations are chosen by the engineer or the design codes, and some are selected by the optimization procedure to be presented. The set of actual values associated with the expectations chosen by the engineer will be denoted by the vector

$$\boldsymbol{\eta} = (h_1, q, \gamma_c, \gamma_s, \gamma_{st}, f_c, f_y, k_a, k_p, \sigma_{soil}, \mu_{crit}, \tau_{max}, r, \phi_\ell, \phi_t, s_\ell, s_t, c_c, c_{st}, c_t, c_{ex}),$$

and the set of actual values associated with the design variables, chosen by the optimization procedure, will be denoted by the vector

$$\boldsymbol{d} = (a, b, c, d, z_t, h_2, h_3, A_{stem}, A_{toe}, A_{heel}).$$

The corresponding mean or characteristic vectors will be denoted $\bar{\boldsymbol{\eta}}$ and $\bar{\boldsymbol{d}}$, respectively.

Assume that the following failure modes are considered: Sliding, overturning, bearing capacity, and stem, toe and heel flexural and shear failures.

Since, including the three flexural and three shear, we have 9 different failure modes, we define the set I_f of failure modes as

$$I_f = \{s, t, b, stem, toe, heel, sstem, stoe, sheel\}$$

The analysis of these failures can be performed by considering

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) = F_i; \quad i \in I_f \tag{3}$$

where $g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}})$ is the ratio of two opposing magnitudes, such as stabilizing to overturning forces, strengths to ultimate stresses, etc., and the stability coefficients, F_i, are random quantities. The wall will be safe if and only if

$$F_s, F_t, F_b, F_{stem}, F_{toe}, F_{heel}, F_{sstem}, F_{stoe}, F_{sheel} \geq 1.$$

Note also that only the usually relevant failure modes have been considered, but other failures are also possible, such as shallow or deep soil failures; however, for the sake of clarity, these have been ignored. It is important to mention that the serviceability states are not considered here, but the ultimate limit states.

4 The proposed method

In this section we describe the proposed method that not only allows a perfect dialog between classical and probability-based designers, but also a double safety check.

Note that the probabilities of failure are very sensitive to joint probabilities and tail assumptions (see Galambos [13] and Castillo [5]), and therefore the safety factors allow a correction to be made when this occurs in the unsafe direction.

When several modes of failure are considered (overturning, sliding, bending, etc.), the calculation of the global failure probability is too complicated, because it involves dealing with the union of several sets (those resulting from each of the failure modes) and hence its boundary is highly irregular (non-differentiable). To avoid this problem, and make it possible for safety factors and failure probabilities to coexist, we present here a method that bases the design on fixing bounds for the safety factors and probabilities for each failure mode, instead of fixing a global failure probability. This leads to a design such that the corresponding probabilities of failure against each mode and the associated safety factors are used to guarantee the safety of the engineering work. This solution can satisfy both types of engineers, those who like working with safety factors, and those desiring failure probabilities. The resulting design is a combination of both, though it can be either of the two, if the other set of constraints is less strict. It can also shed some light on the conservative or non-conservative character of the design.

Though it is fully acceptable to propose a method for individual failure modes and not for the system, an accurate enough upper bound for the failure probability of the system, based on those for the modes, can be easily calculated.

Thus, the proposed method

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimizes}} \quad h(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \tag{4}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \geq F_i^k; \quad i \in I_f, \tag{5}$$

and

$$\beta_{F_i}(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \geq \beta_i^0; \quad i \in I_f. \tag{6}$$

where β_{F_i}; $i \in I_f$ are the Cornell's reliability indices associated with all failure modes.

Since in order to evaluate each of the constraints (6) one needs to solve a minimization problem, the problem (4)-(6) cannot be solved directly. In other words, the optimization procedure and the reliability constraints are coupled. A similar coupling was already mentioned, in a different approach, by Royset, Der Kiureghian and Polak [26] who solved it using an interesting procedure. Here, a new procedure is presented that consists of a sequence of classical designs that minimize the cost. In each step, exact values of the actual safety factors, and exact values or bounds for the probabilities of failure for the different modes are calculated, and the corresponding safety factors updated, until the resulting design satisfies both the required safety factors and failure probability bounds. As a result, the engineer is informed about the failure probabilities for the different modes, as required by modern analysis, and the corresponding safety factors, as in the classical analysis.

An advantage of this approach is that the optimization procedure and the reliability calculations are decoupled.

The method has an initialization part:

Initialization. The safety factors bounds $F_i^1; i \in I_f$ are initialized to their required lower bounds $F_i^0; i \in I_f$

and then its three main parts are repeated until convergence (iteration k is described below):

Part 1. Optimal classical design. This consists of an optimal classical design based on the actual safety factors bounds:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \quad h(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \tag{7}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \geq F_i^k; \;\; i \in I_f. \tag{8}$$

As a result of this process, we obtain the corresponding optimal mean values ($\bar{\boldsymbol{d}}^k$) for the design variables. This is the design a classical designer will choose for minimizing the cost given the safety factor bounds $F_i^k; \;\; i \in I_f$.

Part 2. Evaluation of failure probabilities. The probabilities of failure, $P_i^k = q(\bar{\boldsymbol{d}}^k, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}); \;\; i \in I_f$, or their corresponding β-values, $\beta_i^k = s(\bar{\boldsymbol{d}}^k, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}); \;\; i \in I_f$, associated with all failure modes are evaluated or their upper bounds are determined based on the values of the design variables obtained in Part 1. This involves the solution of one optimization problem per failure mode.

As is well known, the calculation of the probabilities of failure associated with each failure mode ($i \in I_f$) can be performed solving the following non linear programming problem

$$\underset{\mathbf{z}}{\text{Minimize}} \quad \beta_i = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{9}$$

subject to

$$\begin{aligned} z_j &= h_j(\boldsymbol{d}_i, \boldsymbol{\eta}_i; \bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}); \; j = 1, \ldots, n & (10) \\ g_i(\boldsymbol{d}_i, \boldsymbol{\eta}_i) &= 1 & (11) \end{aligned}$$

where $z_j; j = 1, 2, \ldots, n$ are the usual independent standard normal random variables. As a result we obtain the reliability index β_i^k and the design point or point of maximum likelihood $(\boldsymbol{d}_i^k, \boldsymbol{\eta}_i^k)$ for mode i at iteration k.

Once the optimal value of the objective function β_i has been obtained, we can use the well-known FORM/SORM formulas to calculate P_{f_i}.

Part 3. Updating safety factors. At iteration k, the safety factor bounds are adequately updated for the actual safety factors and the failure probabilities to satisfy the required bounds:

$$F_i^k = \max(F_i^{k-1} + \Delta F_i^{k-1}, F_i^0) = \max(F_i^{k-1} + \rho(\beta_i^0 - \beta_i^k), F_i^0); \;\; i \in I_f, \tag{12}$$

where $\beta_i^0; i \in I_f$ are the desired β lower bounds (associated with probability bounds), and ρ is a small positive constant.

If, after updating, some safety factor F_i^k becomes smaller than the associated lower bound for F_i^0, it is kept equal to F_i^0 (this explains the use of the max function in (12)).

The final result of the above procedure is an optimal classical design ($\bar{\boldsymbol{d}}^*$) with the resulting safety factors, which is at the same time an optimal probability-based design, as it satisfies the probability requirements.

Note that the actual safety factors need to be calculated using expressions $g_i(\bar{\boldsymbol{d}}^*, \bar{\boldsymbol{\eta}})$, because the values F_i^k are only bounds (not necessarily active).

The idea of combining both approaches has clear advantages, since

1. There is a double security check.
2. If the required safety factors bounds are too low or too high, the failure probability method warns the engineer about this.
3. If the failure probability constraints are extreme or the probability assumptions unrealistic, the associated safety factors warn the engineer about this problem, and allow a trial and error adjustement procedure until an agreement between the safety factors and the probability bounds is obtained.
4. Since the tail probabilities are known to be very influential in the final design, the safety factors can be used to test these sensitivities.

4.1 The Safety Factor-Probability Method Algorithm

The above methodology can be summarized in the following algorithm.

Algorithm 1 (Design with safety factors and probability constraints)

- **Input:** *The lower bounds* $\{F_1^0, F_2^0, \dots, F_m^0\}$ *for the safety factors* $\{F_1, F_2, \dots, F_m\}$*, the lower bounds* $\{\beta_1^0, \beta_2^0, \dots, \beta_m^0\}$ *for the* β*-values* $\{\beta_1, \beta_2, \dots, \beta_m\}$*, with respect to all failure modes, and an error value* ϵ *to control convergence of the procedure.*
- **Output:** *An optimal design defined by the mean values of the design variables and actual safety factors, and probabilities of failure for the different failure modes.*

Initialization. *Initiate the safety factor bounds* $\{F_1^1, F_2^1, \dots, F_m^1\}$ *to their required lower bounds* $\{F_1^0, F_2^0, \dots, F_m^0\}$.

Step 1: Master problem *The optimal classical design minimizing the cost subject to the safety factor constraints is obtained.*

Step 2: Subproblems *Exact values or upper bounds for the probabilities of failures or* β*-values for all failure modes are calculated, solving a minimization problem per failure mode (level II or III approach).*

Step 3: Updating bounds *Safety factors bounds are updated using Formula (12).*

Step 4: Checking convergence *If changes in the design variables in the current iteration are larger than a given threshold value* ϵ*, go to Step 1. Otherwise, go to Step 5.*

Step 5: Post process *Calculate the actual safety factors associated with non-active constraints. Stop and return design values, associated safety factors, and probabilities of failure for the different failure modes.*

5 Sensitivity analysis

A sensitivity analysis improves the quality of any study. Under a sensitivity analysis, not only the solution of the problem is sought but also how sensitive it is to data changes.

The sensitivity analysis is not a standard procedure and is very useful to (a) the designer, who can know which data values are more influential on the design, (b) to the builder, who can know how changes in prices influence the total cost, and (c) to the code maker, who can know the costs and reliability changes associated with an increase or decrease in the required safety factors or failure probabilities. The methodology proposed below is very simple, efficient and allows all the sensitivities to be calculated simultaneously. At the same time it is the natural way of evaluating sensitivities when optimization procedures are present.

A sensitivity analysis can be easily performed using the fact that almost all mathematical programming software packages give the values of the dual variables. To this end, we need to convert the data values into artificial variables and add the corresponding constraints. In other words, we transform our initial problem:

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimize}} \quad h(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \tag{13}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \geq F_i; \;\; i \in I_f. \tag{14}$$

into the equivalent problem:

$$\underset{\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*}{\text{Minimize}} \quad h(\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*) \tag{15}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \boldsymbol{\eta}^*) \geq F_i; \;\; i \in I_f. \tag{16}$$

and

$$\boldsymbol{\eta}^* = \bar{\boldsymbol{\eta}} \tag{17}$$

and the problem:

$$\underset{\mathbf{z}}{\text{Minimize}} \quad \beta(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}) = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{18}$$

subject to

$$z_j = h_j(\boldsymbol{d}, \boldsymbol{\eta}; \bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}); \; j = 1, \ldots, n \tag{19}$$

$$g(\boldsymbol{d}, \boldsymbol{\eta}) = 1 \tag{20}$$

into the equivalent problem:

$$\underset{\mathbf{z}, \boldsymbol{d}^*, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*}{\text{Minimize}} \quad \beta(\boldsymbol{d}^*, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*) = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{21}$$

subject to

$$z_j = h_j(\boldsymbol{d}, \boldsymbol{\eta}; \boldsymbol{d}^*, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*); \; j = 1, \ldots, n \tag{22}$$

$$g(\boldsymbol{d}, \boldsymbol{\eta}) = 1 \tag{23}$$

and

$$(\boldsymbol{d}^*, \boldsymbol{\eta}^*, \boldsymbol{\kappa}^*) = (\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}). \tag{24}$$

However, when solving the second problems, which have exactly the same solutions as their initial problems, one can obtain the values of the dual variables associated with the constraints in (17) or (24). These are the sensitivities of the objective function with respect to the parameters $\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}$ and $\boldsymbol{\kappa}$, i.e. they indicate how much the objective function changes with a very small unit increment of the corresponding parameter.

6 The Retaining Wall Example

In this section we describe the application of the proposed methods to the retaining wall example. Detailed treatment of the constraints are given in the Appendix.

6.1 Distributional assumptions of the model

A wall design at levels II and III implies defining the random properties of all the variables involved. In this example, all the design, geometric, acting agent and material definition data are assumed to be normal and independent random variables which means are the corresponding design values denoted by an overbar placed on the corresponding variables and standard deviations shown in Table 1.

For the sake of simplicity, in this paper the following set of variables is assumed to be deterministic:

$$\{r, \phi_\ell, \phi_t, s_\ell, s_t, c_c, c_{st}, c_t, c_{ex}\}$$

Table 1: Distributional assumptions for the variables in the illustrative example ($\boldsymbol{\kappa}$ vector values).

Type	Variable	Standard deviation	Type	Variable	Standard deviation
Geometric variables	a	$0.025m$	Strength variables	f_c	$3MPa$
	b	$0.025m$		f_y	$22MPa$
	c	$0.025m$		σ_{soil}	$0.25MPa$
	d	$0.025m$		τ_{max}	$0.04MPa$
	z_t	$0.025m$	Unit weight variables	γ_c	$1kN/m^3$
	h_1	$0.025m$		γ_s	$2kN/m^3$
	h_2	$0.025m$		γ_{st}	$0.2kN/m^3$
	h_3	$0.025m$	Other variables	k_a	0.05
	A_{stem}	$0.00001m^2$		k_p	0.1
	A_{toe}	$0.00001m^2$		μ_{crit}	0.05
	A_{heel}	$0.00001m^2$		q	$8kN/m$

6.2 Numerical solution and discussion

The proposed method was implemented in GAMS (General Algebraic Modeling System) (see Castillo, Conejo, Pedregal, García and Alguacil [6]) for the wall example. Suppose we want to design a retaining wall with the following assumptions:

$$
\begin{array}{lllll}
\bar{h}_1 = 5m; & \bar{\gamma}_c = 25kN/m^3; & \bar{\gamma}_s = 20kN/m^3; & \bar{f}_c = 30MPa; & \bar{f}_y = 450MPa; \\
\bar{k}_a = 0.5; & \bar{k}_p = 3; & \bar{\mu}_{crit} = 0.5; & \bar{\sigma}_{soil} = 0.25MPa; & \bar{q} = 40kN/m; \\
r = 0.05m; & F_t^0 = 1.2; & F_s^0 = 1.15; & F_b^0 = 1.2; & F_{stem}^0 = 1.4; \\
F_{toe}^0 = 1.4; & F_{heel}^0 = 1.4; & F_{sstem}^0 = 1.2; & F_{stoe}^0 = 1.2; & F_{sheel}^0 = 1.2; \\
c_c = 66/m^3; & c_{st} = 0.06\text{euro}/N; & c_t = 12\text{euro}/m^3; & c_{ex} = 3.6\text{euro}/m^3; & \tau_{max} = 0.4MPa; \\
\phi_\ell = 0.02m; & \phi_t = 0.02m; & s_\ell = 0.2m; & s_t = 0.2m; &
\end{array}
$$

where these are the data values that have been chosen by the designer.

The proposed method was used for the design using these data. Table 2 shows the convergence of the process. The last column of the table shows the design values of $a, b, c, d, h_2, h_3, z_t, A_{stem}, A_{toe}$ and A_{heel}, together with the safety factors and associated β-values. The design was made to guarantee values of the safety factors $F_t \geq 2.0; F_s \geq 2.0; F_b \geq 2.0; F_{stem} \geq 1.3; F_{toe} \geq 1.3; F_{heel} \geq 1.3; F_{sstem} \geq 1.4; F_{stoe} \geq 1.4; F_{sheel} \geq 1.4$ and $\beta_t, \beta_s, \beta_b, \beta_{stem}, \beta_{toe}, \beta_{heel}, \beta_{sstem}, \beta_{stoe}$ and β_{sheel} larger than or equal to $\beta_i^0 = 3.71$. The active values appear underlined in this table.

The following conclusions can be drawn from Table 2.

1. The process converges in 24 iterations, but practically the same results are obtained after iteration 15.

2. The safety factor bound F_s^0 is active, i.e. it leads to a stricter constraint than the corresponding reliability index bound. Note that the corresponding safety index β_s is greater than $\beta_s^0 = 3.71$.

3. The reliability indices bounds $\beta_b^0, \beta_{stem}^0, \beta_{toe}^0, \beta_{heel}^0, \beta_{sstem}^0, \beta_{stoe}^0, \beta_{sheel}^0$ are active, i.e. they lead to stricter constraints than the corresponding safety factors.

4. The safety factor F_t^0 and its corresponding reliability index bound β_t^0 are both inactive. Note that both the safety factor and the beta value are greater than their corresponding bounds. Restrictions associated with the other failure modes make this one safe enough.

5. The optimal values for d and z_t are $d = 0.3$ and $z_t = 0.3$. This implies that the constraints (25) (see Appendix) are active. In other words, a lower cost can be obtained if these constraints are removed, but then, the optimal solutions would be $d = z_t = 0$.

Table 2: Illustration of the iterative procedure. The design and final values are boldfaced.

		ITERATIONS				
Variable	Units	1	5	15	20	24 (end)
Cost	*euro*	1105.2	1219.3	1275.4	1280.1	**1281.6**
h_1	m	5.00	5.00	5.00	5.00	**5.00**
z_t	m	0.30	0.30	0.30	0.30	**0.30**
a	m	1.55	1.70	1.79	1.80	**1.81**
b	m	0.90	1.10	1.21	1.22	**1.23**
c	m	4.83	4.29	4.01	3.98	**3.97**
d	m	0.30	0.30	0.30	0.30	**0.30**
h_2	m	0.50	0.58	0.61	0.62	**0.62**
h_3	m	1.04	1.34	1.50	1.51	**1.51**
A_{stem}	m^2	0.00166	0.00165	0.00160	0.00159	**0.00159**
A_{toe}	m^2	0.00044	0.00041	0.00038	0.00037	**0.00037**
A_{heel}	m^2	0.00180	0.00162	0.00145	0.00143	**0.00143**
F_t	–	6.45	6.09	5.96	5.96	**5.96**
F_s	–	2.00	2.00	2.00	2.00	**2.00**
F_b	–	2.00	2.00	2.01	2.01	**2.01**
F_{stem}	–	1.30	1.68	1.84	1.86	**1.86**
F_{toe}	–	1.30	1.49	1.50	1.50	**1.50**
F_{heel}	–	1.30	1.70	1.87	1.88	**1.89**
F_{sstem}	–	1.40	1.73	1.91	1.93	**1.93**
F_{stoe}	–	1.40	1.67	1.80	1.81	**1.82**
F_{sheel}	–	1.40	1.72	1.91	1.93	**1.93**
β_t	–	–	24.75	24.17	24.13	**24.13**
β_s	–	–	3.88	3.81	3.80	**3.80**
β_b	–	–	3.72	3.71	3.71	**3.71**
β_{stem}	–	–	3.10	3.67	3.70	**3.71**
β_{toe}	–	–	3.61	3.73	3.72	**3.71**
β_{heel}	–	–	3.06	3.67	3.70	**3.71**
β_{sstem}	–	–	3.11	3.65	3.70	**3.71**
β_{stoe}	–	–	3.25	3.67	3.70	**3.71**
β_{sheel}	–	–	3.11	3.64	3.69	**3.71**

6.3 Sensitivity analysis

The sensitivities for the illustrative wall example are given in Tables 3 and 4. Table 3 gives the cost sensitivities associated with the optimal classical design. It allows us to know how much the total cost of the wall changes when a small change in a single data value is made. This information is extremely useful during the construction process for controlling the cost, and for analyzing how the changes in the safety factors required by the codes influence the total cost of engineering works. For example, a change of one euro in the unit cost c_c of the concrete leads to a cost increase of 11.511euros (see the corresponding entry in Table 3). Similarly, an increase in the safety factor lower bound F_t^0 does not change the cost, but an increase in the safety factor lower bound F_s^0 increases the cost by 193.43 euros per unit of increase.

Table 4 gives the sensitivities associated with the β-values in adimensional form. It is useful to know how much the corresponding β-value changes when a small change in a single data value is made, for example, the means, standard deviations, etc. In this table the designer can easily analyze how the quality of the material (reduced standard deviations in f_c or f_y) or precision in the construction of the work (reduced standard deviations in h_1, h_2 and b) influence the safety of the wall.

Note that all sensitivities with respect to standard deviations are null or negative as expected (the larger

Table 3: Cost sensitivities with respect to the data values in the wall illustrative example.

c_c	c_t	c_{st}	c_{ex}	r	ϕ_l
11.511	12.811	2730.090	56.743	738.238	3.667
ϕ_t	s_l	s_t	F_t^0	F_s^0	F_b^0
5.243	-110.025	-157.291	0.000	193.430	55.728
F_{stem}^0	F_{toe}^0	F_{heel}^0	F_{sstem}^0	F_{stoe}^0	F_{sheel}^0
27.349	8.287	24.936	1.427	159.455	130.364
$\bar{\gamma}_c$	$\bar{\gamma}_s$	$\bar{\gamma}_{st}$	$\bar{f}_c$	$\bar{f}_y$	$\bar{\sigma}_{soil}$
-5.595	22.042	2.087	-0.024	-0.245	-448.968
$\bar{\tau}_{max}$	$\bar{\mu}_{crit}$	$\bar{q}$	$\bar{k}_a$	$\bar{k}_p$	$\bar{h}_1$
-1360.752	-773.721	11.666	2492.194	-18.392	490.227

the dispersion, the smaller the reliability.

7 General conclusions

A new method has been presented for designing engineering works that presents the following advantages:

1. Since safety factors and probabilities of failure are dealt with, the method allows communication between classical and probability-based designers.
2. The proposed method takes full advantage of the optimization packages, in the sense that:
   (a) The constraints need not be written in terms of the design variables. Auxiliary or intermediate variables can be used.
   (b) The cost function and the constraints need not be written in explicit form, i.e. auxiliary variables and equations can be used to facilitate the statement of the problem.
   (c) The failure region need not be written in terms of the normalized (transformed) variables. The transformation equation, in direct or inverse form, is sufficient.
   (d) The responsibility for iterative methods is given to the optimization software.
3. Sensitivity values are given, for free, if one converts the data values into artificial variables, by printing the values of the dual problem. This allows us determine how much a small change in any of the data values, such as the cost of the materials, the safety factors, etc., affects the total cost and the reliability indices of the engineering work.
4. The method controls for safety against all failure modes by a double check: via safety factors and via reliability indices.
5. The resulting designs are:
   (a) Automated
   (b) Optimal.
   (c) Designer independent.

Table 4: Sensitivities $\bar{x}\frac{\partial \beta_i}{\partial x}$, with respect to the data values x, for the illustrative wall example (only non-zero values are shown).

Data x	$\bar{x}\frac{\partial\beta_t}{\partial x}$	$\bar{x}\frac{\partial\beta_s}{\partial x}$	$\bar{x}\frac{\partial\beta_b}{\partial x}$	$\bar{x}\frac{\partial\beta_{stem}}{\partial x}$	$\bar{x}\frac{\partial\beta_{toe}}{\partial x}$	$\bar{x}\frac{\partial\beta_{heel}}{\partial x}$	$\bar{x}\frac{\partial\beta_{sstem}}{\partial x}$	$\bar{x}\frac{\partial\beta_{stoe}}{\partial x}$	$\bar{x}\frac{\partial\beta_{sheel}}{\partial x}$
r	0.00	0.00	0.00	-0.30	-0.62	-0.36	-0.23	-0.36	-0.28
μ_{h_1}	-30.69	-4.73	-7.75	-16.32	-16.32	-15.60	-7.81	-9.76	-9.15
μ_a	8.42	-0.11	3.42	0.00	-14.50	3.92	0.00	0.06	1.36
μ_b	6.55	0.47	1.66	8.58	2.86	2.93	5.65	1.65	1.35
μ_c	25.78	4.23	2.70	0.00	-2.11	-4.77	0.00	1.39	0.48
μ_d	0.24	0.15	-0.16	0.10	-0.60	0.11	0.00	-0.22	0.14
μ_{h_2}	-4.00	-0.56	-0.78	0.00	11.39	1.48	0.00	2.77	1.48
μ_{h_3}	4.17	2.04	-1.13	0.00	6.31	4.46	0.00	1.75	2.66
μ_{z_t}	-1.83	0.05	-0.55	-0.97	-0.48	-0.91	-0.37	-0.43	-0.47
$\mu_{A_{stem}}$	0.00	0.00	0.00	6.75	0.00	0.00	0.00	0.00	0.00
$\mu_{A_{toe}}$	0.00	0.00	0.00	0.00	9.59	0.00	0.00	0.00	0.00
$\mu_{A_{heel}}$	0.00	0.00	0.00	0.00	0.00	6.63	0.00	0.00	0.00
μ_{γ_c}	6.13	2.16	-1.17	0.68	-3.38	0.70	0.00	-1.06	0.74
μ_{γ_s}	3.73	-0.59	-1.37	-2.73	-2.15	-2.66	-2.57	-1.99	-2.90
μ_{f_c}	0.00	0.00	0.00	0.09	0.12	0.05	0.00	0.00	0.00
μ_{f_y}	0.00	0.00	0.00	7.20	10.29	7.07	0.00	0.00	0.00
$\mu_{\sigma_{soil}}$	0.00	0.00	7.26	0.00	0.00	0.00	0.00	0.00	0.00
$\mu_{\tau_{max}}$	0.00	0.00	0.00	0.00	0.00	0.00	7.40	8.06	7.47
$\mu_{\mu_{crit}}$	0.00	7.55	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_q	-1.89	-1.28	-1.49	-3.09	-2.65	-3.06	-1.97	-1.91	-2.21
μ_{k_a}	-7.22	-5.92	-3.51	-6.12	-2.37	-6.35	-4.63	-3.81	-3.89
μ_{k_p}	0.01	0.38	0.09	0.00	0.37	0.10	0.05	0.08	0.08
σ_{h_1}	-0.57	0.00	-0.01	-0.02	-0.02	-0.02	-0.01	-0.01	-0.01
σ_{z_t}	-0.56	0.00	-0.01	-0.02	-0.01	-0.02	0.00	0.00	-0.01
σ_a	-0.33	0.00	-0.01	0.00	-0.15	-0.01	0.00	0.00	0.00
σ_b	-0.43	0.00	0.00	-0.11	-0.01	-0.01	-0.05	0.00	0.00
σ_c	-0.63	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
σ_d	-0.01	0.00	0.00	0.00	-0.01	0.00	0.00	0.00	0.00
σ_{h_2}	-0.64	0.00	0.00	0.00	-0.79	-0.01	0.00	-0.05	-0.01
σ_{h_3}	-0.11	0.00	0.00	0.00	-0.04	-0.02	0.00	0.00	-0.01
$\sigma_{A_{stem}}$	0.00	0.00	0.00	-0.01	0.00	0.00	0.00	0.00	0.00
$\sigma_{A_{toe}}$	0.00	0.00	0.00	0.00	-0.24	0.00	0.00	0.00	0.00
$\sigma_{A_{heel}}$	0.00	0.00	0.00	0.00	0.00	-0.01	0.00	0.00	0.00
σ_{γ_c}	-1.45	-0.03	-0.01	0.00	-0.07	0.00	0.00	-0.01	0.00
σ_{γ_s}	-3.36	-0.01	-0.07	-0.28	-0.17	-0.26	-0.25	-0.15	-0.31
σ_{f_y}	0.00	0.00	0.00	-0.46	-0.94	-0.44	0.00	0.00	0.00
$\sigma_{\sigma_{soil}}$	0.00	0.00	-2.82	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma_{\tau_{max}}$	0.00	0.00	0.00	0.00	0.00	0.00	-2.03	-2.41	-2.07
$\sigma_{\mu_{crit}}$	0.00	-2.16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
σ_q	-3.46	-0.25	-0.33	-1.41	-1.05	-1.39	-0.58	-0.54	-0.72
σ_{k_a}	-12.58	-1.33	-0.46	-1.39	-0.21	-1.50	-0.79	-0.54	-0.56

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A Appendix

This appendix gives the details of the intermediate equations needed to deal with the constraints and the cost function.

A.1 Forces acting on the wall

Before deriving the set of constraints, the total weight of the wall and soil, and the earth pressures acting on the wall are determined together with the corresponding points of application. The constraints are then established. To this end, intermediate or auxiliary variables that facilitate the work can be used.

Note that optimization programs allow the use of these auxiliary variables, and that an explicit expression for the constraints in terms of the design variables is not needed. This is an important advantage, as such a process is complicated.

Total weight of the wall and soil. Figure 3(a), illustrates the weights, $w_i; i = 1, 2, 3, 4$, of the different pieces of the wall, and their respective locations. Similarly, the weights, $s_i; i = 1, 2, 3$, of the different pieces of the soil are illustrated in Figure 3(b).

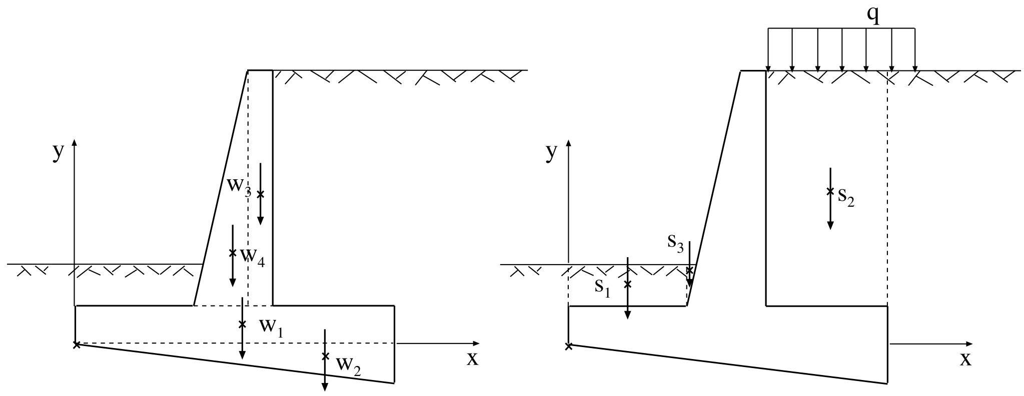
Earth pressures. Figure 4(a) shows the soil pressures.

A.2 Design criteria and safety factors requirements

Two design constraints are:

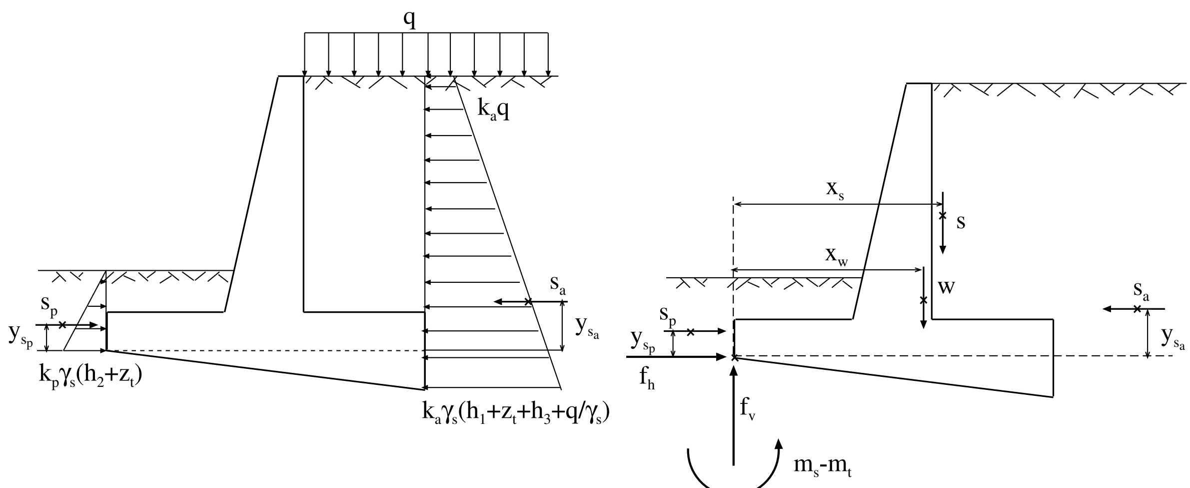
$$d \geq 0.3; \quad z_t \geq 0.3 \tag{25}$$

and the safety factors constraints:



(a) Weight forces (b) Soil forces

Figure 3: Decomposition of the cross section of the cantilever wall into simpler geometrical pieces (triangles and rectangles) to calculate: (a) the weights of the concrete, and (b) the soil weights.



(a) (b)

Figure 4: (a) Illustration of the earth pressures on both sides of the wall. (b) Weight and soil pressure forces acting on the wall.

Overturning constraint:

$$m_s/m_t \geq F_t \tag{26}$$

where m_s and m_t are shown in Figure 4(b).

Sliding constraint:

$$\mu_{crit}/\mu \geq F_s \tag{27}$$

where μ is the actual friction factor.

Bearing capacity constraint:

$$\frac{\sigma_{soil}}{\sigma_i} \quad \geq \quad F_b \tag{28}$$

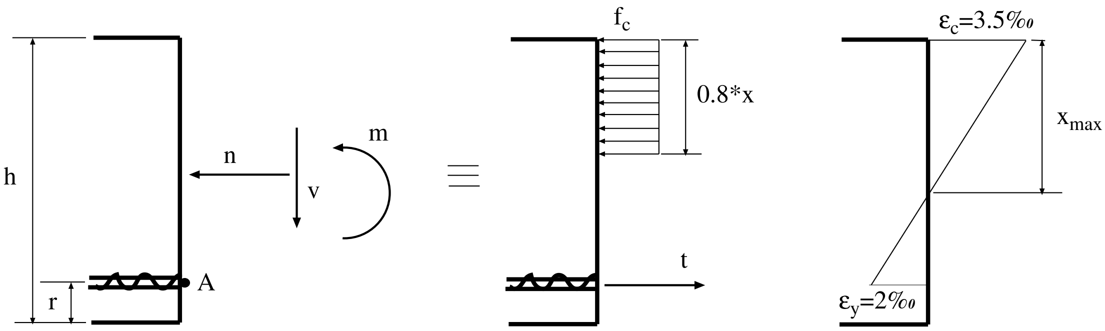


Figure 5: Illustration of the calculation of the steel reinforcing bars.

where σ_i is the maximum stress at the left end of the toe.

Flexural and shear constraints at the heel, toe and stem:
First of all, the reinforcement steel needed in a general section subject to a normal force (n), a shear force (v) and a bending moment (m) (see Figure 5) are calculated:

$$\begin{array}{rrcl} \sum M_A = 0: & f_c 0.8x(h-r-0.4x) & = & m + n(h/2 - r) \\ \sum F_H = 0: & t & = & f_c 0.8x - n \\ & \dfrac{A_s f_y}{t} & \geq & F_k; \;\; k = stem, toe, heel \\ & \dfrac{\tau_{max}}{v/(h-r)} & \geq & F_j; \;\; j = sstem, stoe, sheel \end{array} \quad (29)$$

A.3 Function to be optimized

Since this is only an illustrative example of the proposed method, for the sake of simplicity the life span of the wall, maintenance or reparation costs are not considered. Then, the total cost of the wall is:

$$Cost = h(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) = v_c c_c + s_t c_t + w_{st} c_{st} + v_{ex} c_{ex}$$

where the required total volume of concrete, v_c, the total timber surface, s_t, the total weight of steel, w_{st}, and the total excavation volume v_{ex} are:

$$\begin{array}{rcl} v_c & = & w/\gamma_c \\ s_t & = & h_2 + h_3 + h_1 + z_t + \sqrt{(b-d)^2 + h_1^2} \\ w_{st} & = & A_1(h_1 + z_t + c_{aux2} + 0.5)\gamma_{st} + A_2(d_{aux} + d_{aux2} + d_{aux3})\gamma_{st} + A_3(a+b+c)\gamma_{st} \\ & & +(h_2 + h_3 + d + \sqrt{(b-d)^2 + (h_1+z_t)^2} + 4)\pi\phi_t^2\gamma_{st}/(4s_t) + \\ & & (\sqrt{(b-d)^2 + (h_1+z_t)^2} + d + h_1 + z_t + c + h_3 + d_{aux} + d_{aux2} + d_{aux3} + h_2 + a)\phi_l^2\gamma_{st}/(4s_l) \\ v_{ex} & = & v_c + (h_1 + z_t)c + z_t a + (h_1 + z_t + h_3)^2/2 + (h_2 + z_t)^2/2 \end{array} \quad (30)$$

where $\gamma_{st} = 78.5 kN/m^3$.