# Sensitivity analysis in Optimization and Reliability Problems.

Enrique Castillo[1], Roberto Mínguez[2], Carmen Castillo[3]

**Abstract**

The paper starts giving the main results that allow a sensitivity analysis to be performed in a general optimization problem, including sensitivities of the objective function, the primal and the dual variables with respect to data. In particular, general results are given for non-linear programming, and closed formulas for linear programming problems are supplied. Next, the methods are applied to a collection of civil engineering reliability problems, which includes a bridge crane, a retaining wall and a vertical breakwater. Finally, the sensitivity analysis formulas are extended to calculus of variations problems and a slope stability problem is used to illustrate the methods.

**Key Words:** Failure modes, nonlinear programming, optimization, probabilistic models, reliability analysis, safety factors, sensitivity analysis, bridge crane design, retaining wall, rubble-mound breakwaters, calculus of variations, slope stability.

# 1 Introduction

This paper deals with sensitivity analysis. Sensitivity analysis discusses "how" and "how much" changes in the parameters of an optimization problem modify the optimal objective function value and the point where the optimum is attained (see Castillo et al. (1996)).

Today, it is not enough to give users the solutions to their problems. In addition, they require knowledge of how these solutions depend on data and/or assumptions. Therefore, data analysts must be able to supply the sensitivity of their conclusions to model and data. Sensitivity analysis allows the analyst to assess the effects of changes in the data values, to detect outliers or wrong data, to define testing strategies, to increase the reliability, to optimize resources, reduce costs, etc.

Sensitivity analysis increases the confidence in the model and its predictions, by providing an understanding of how the model responds to changes in the inputs. Adding a sensitivity analysis to an study means adding extra quality to it.

Sensitivity analysis is not a standard procedure, however, it is very useful to (a) the designer, who can know which data values are the most influential on the design, (b) to the builder, who can know how changes in the material properties or the prices influence the total reliability or cost of the work being designed, and (c) to the code maker, who can know the costs and reliability implications associated with changes in the safety factors or failure probabilities. The methodology proposed below is very simple, efficient and allows all the sensitivities to be calculated simultaneously. At the same time it is the natural way of evaluating sensitivities when optimization procedures are present.

The paper is structured as follows. In Section 2 the statement of optimization problems and the conditions to be satisfied are presented. Section 3 gives the formula to get sentitivities with respect to the objective function. In Section 4 a general method for deriving all possible sensitivities is given. Section 5 deals with some examples and the interpretation of the sensitivity results. In Section 6 the methodology is extended to calculus of variations, and finally, Section 7 provides some relevant conclusions.

[1]Full Professor, Department of Applied Mathematics and Computational Sciences, University of Cantabria, Avda. Castros s/n., 39005 Santander, Spain. e-mail: castie@unican.es

[2]Assistant Professor, Department of Applied Mathematics, University of Castilla-La Mancha, 13071 Ciudad Real, Spain. e-mail: roberto.minguez@uclm.es

[3]Assistant Professor, Department of Civil Engineering, University of Castilla-La Mancha, 13071 Ciudad Real, Spain. e-mail: mariacarmen.castillo@uclm.es

## 2 Statement of the problem

Consider the following primal non-linear programming problem (NLPP):

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad z_P = f(\boldsymbol{x}, \boldsymbol{a}) \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) &= \boldsymbol{b} : \boldsymbol{\lambda} && (2)\\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) &\leq \boldsymbol{c} : \boldsymbol{\mu}, && (3)\end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $\boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^\ell$, $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ with $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = (h_1(\boldsymbol{x}, \boldsymbol{a}), \ldots, h_\ell(\boldsymbol{x}, \boldsymbol{a}))^T$, $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) = (g_1(\boldsymbol{x}, \boldsymbol{a}), \ldots, g_m(\boldsymbol{x}, \boldsymbol{a}))^T$ are regular enough for the mathematical developments to be valid over the feasible region $S(\boldsymbol{a}) = \{\boldsymbol{x} | \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = \boldsymbol{b}, \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) \leq \boldsymbol{c}\}$, $f, \boldsymbol{h}, \boldsymbol{g} \in C^2$, and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the vectors of dual variables. It is also assumed that the problem (1)–(3) has an optimum at $\boldsymbol{x}^*$.

Any primal problem $P$, as that stated in (1)–(3), has an associated dual problem $D$, which is defined as:

$$\underset{\boldsymbol{\lambda}, \boldsymbol{\mu}}{\text{Maximize}} \; z_D = \underset{\boldsymbol{x}}{\text{Inf}} \; \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \tag{4}$$

subject to

$$\boldsymbol{\mu} \geq \boldsymbol{0}, \tag{5}$$

where

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = f(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\lambda}^T(\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{b}) + \boldsymbol{\mu}^T(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{c}) \tag{6}$$

is the Lagrangian function associated with the primal problem (1)–(3), and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the dual variables, are vectors of dimensions $\ell$ and $m$, respectively. Note that only the dual variables ($\boldsymbol{\mu}$ in this case) associated with the inequality constraints ($\boldsymbol{g}(\boldsymbol{x})$ in this case), must be nonnegative.

Given some regularity conditions on local convexity (see Luenberger (1984) or Castillo et al. (2005a)), if the primal problem (1)–(3) has a locally optimal solution $\boldsymbol{x}^*$, the dual problem (4)–(6) also has a locally optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, and the optimal values of the objective functions of both problems coincide.

### 2.1 Karush-Kuhn-Tucker conditions

The primal (1)–(3) and the dual (4)–(6) problems, respectively, can be solved using the Karush-Kuhn-Tucker (KKTCs) first order necessary conditions (see, for example, Luenberger (1984), Bazaraa et al. (1993) or Castillo et al. (2001)):

$$\begin{aligned} \nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \boldsymbol{\lambda}^{*T} \nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}) + \boldsymbol{\mu}^{*T} \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) &= \boldsymbol{0} && (7)\\ \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}) &= \boldsymbol{b} && (8)\\ \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) &\leq \boldsymbol{c} && (9)\\ \boldsymbol{\mu}^{*T}(\boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) - \boldsymbol{c}) &= 0 && (10)\\ \boldsymbol{\mu}^* &\geq \boldsymbol{0} && (11)\end{aligned}$$

where $\boldsymbol{x}^*$ and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ are the primal and dual optimal solutions, $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*; \boldsymbol{a})$ is the gradient (vector of partial derivatives) of $f(\boldsymbol{x}^*; \boldsymbol{a})$ with respect to $\boldsymbol{x}$, evaluated at the optimal value $\boldsymbol{x}^*$. The vectors $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*$ are also called the *Kuhn–Tucker multipliers*. Condition (7) says that the gradient of the Lagrangian function in (6) evaluated at the optimal solution $\boldsymbol{x}^*$ must be zero. Conditions (8)–(9) are called *the primal feasibility* conditions. Condition (10) is known as the *complementary slackness condition*. Finally, condition (11) requires the nonnegativity of the multipliers of the inequality constraints, and is referred to as the *dual feasibility conditions*.

Note that in the present analysis the regular non-degenerate case is only considered, which is the most frequent in real life applications. To understand the meaning of these assumptions the following definitions are given.

**Definition 1 (Regular point)** *The solution $\boldsymbol{x}^*$ of the optimization problem (1)–(3) is said to be a regular point of the constraints if the gradient vectors of the active constraints are linearly independent.*

**Definition 2 (Degenerate inequality constraint)** *An inequality constraint is said to be degenerate if it is active and the associated $\mu$-multiplier is null.*

Once the optimal solution is known, degeneracy can be identified and possibly eliminated. The degenerate case is extensively analyzed in Castillo et al. (2005a).

### 2.2 Some sensitivity analysis questions

When dealing with the optimization problem (1)–(3), the following questions regarding sensitivity analysis are of interest:

1. What are the local sensitivities of $z_P^* = f(\boldsymbol{x}^*, \boldsymbol{a})$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$?
2. What are the local sensitivities of $\boldsymbol{x}^*$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$?
3. What are the local sensitivities of $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$?

The answers to these questions are given in the following sections.

## 3 Sensitivities of the Objective function

Calculating the sensitivities of the objective function with respect to data is extremely easy using the following theorem (see Castillo et al. (2005b)).

**Theorem 1** *Assume that the solution $\boldsymbol{x}^*$ of the above optimization problem is a regular point and that no degenerate inequality constraints exists. Then, the sensitivity of the objective function with respect to the parameter $\boldsymbol{a}$ is given by the gradient of the Lagrangian function*

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = f(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\lambda}^T(\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{b}) + \boldsymbol{\mu}^T(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{c}) \tag{12}$$

*with respect to $\boldsymbol{a}$ evaluated at the optimal solution $\boldsymbol{x}^*, \boldsymbol{\lambda}^*$, and $\boldsymbol{\mu}^*$, i.e.*

$$\frac{\partial z_P^*}{\partial \boldsymbol{a}} = \nabla_{\boldsymbol{a}} \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), \tag{13}$$

Note that this theorem is very useful from the practical point of view because it allows you to know how much the objective function value $z_P^*$ changes when parameters $\boldsymbol{a}$ change.

Note that the sensitivities with respect to $\mathbf{b}$ and $\mathbf{c}$ are their respective gradients.

**Example 1 (Objective function sensitivity with respect to right hand side parameters)** *Consider the optimization problem (1)–(3). Using Theorem 1, i.e., differentiating (12) with respect to $\boldsymbol{b}$ and $\boldsymbol{c}$ one obtains:*

$$\frac{\partial f(\boldsymbol{x}^*, \boldsymbol{a})}{\partial b_i} = -\lambda_i^*; \;\; i = 1, 2, \ldots, \ell; \;\; \frac{\partial f(\boldsymbol{x}^*, \boldsymbol{a})}{\partial c_j} = -\mu_j^*; \;\; j = 1, 2, \ldots, m$$

*i.e., the sensitivities of the optimal objective function value of the problem (1)–(3) with respect to changes in the terms appearing on the right hand side of the constraints are the negative of the optimal values of the corresponding dual variables.*

*For this important result to be applicable to practical cases of sensitivity analysis, the parameters for which the sensitivities are sought must appear on the right hand side of the primal problem constraints.*

At this point the reader can ask him/herself and what about parameters not satisfying this condition, $\boldsymbol{a}$ for example? The answer to this question is given in the following example.

**Example 2 (A practical method for obtaining all the sensitivities of the objective function)** *In this example it is shown how the duality methods can be applied to derive the objective function sensitivities in a straightforward manner. The basic idea is simple. Assume that we desire to know the sensitivity of the objective function to changes in some data values. If we convert the data into artificial variables and set them, by means of constraints, to their actual values, we obtain a problem that is equivalent to the initial optimization problem but has a constraint such that the values of the dual variables associated with them give the desired sensitivities.*

*To be more precise, the primal optimization problem (1)–(3) is equivalent to the following one:*

$$\underset{\boldsymbol{x}, \tilde{\boldsymbol{a}}}{\text{Minimize}} \quad z_P = f(\boldsymbol{x}, \tilde{\boldsymbol{a}}) \tag{14}$$

*subject to*

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{x}, \tilde{\boldsymbol{a}}) &= \boldsymbol{b} : \boldsymbol{\lambda} & (15)\\ \boldsymbol{g}(\boldsymbol{x}, \tilde{\boldsymbol{a}}) &\leq \boldsymbol{c} : \boldsymbol{\mu} & (16)\\ \tilde{\boldsymbol{a}} &= \boldsymbol{a} : \boldsymbol{\eta}. & (17) \end{aligned}$$

*It is clear that problems (1)–(3) and (14)–(17) are equivalent, but for the second the sensitivities with respect to* $\boldsymbol{a}$ *are readily available. Note that to be able to use the important result of Theorem 1, we convert the data* $\boldsymbol{a}$ *into artificial variables* $\tilde{\boldsymbol{a}}$ *and set them to their actual values* $\boldsymbol{a}$ *as in constraint (17). Thus, the negative values of the dual variables* $\boldsymbol{\eta}$ *associated with (17) are the sensitivities sought after, i.e., the partial derivatives* $\partial z_P / \partial a_i = -\eta_i; i = 1, 2, \ldots, p$.

**Example 3 (Linear Programming Case)** *The simplest and very important case where Theorem 1 can be applied is the case of linear programming.*

*Consider the following linear programming problem:*

$$\text{Minimize} \;\; z_P = \sum_{i=1}^{n} c_i x_i \tag{18}$$

*subject to*

$$\begin{aligned} \sum_{i=1}^{n} p_{ji} x_i &= r_j \;:\; \lambda_j; \;\; j = 1, 2, \ldots, \ell & (19)\\ \sum_{i=1}^{n} q_{ki} x_i &\leq s_k \;:\; \mu_k; \;\; k = 1, 2, \ldots, m & (20) \end{aligned}$$

*The Lagrangian function becomes*

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s}) = \sum_{i=1}^{n} c_i x_i + \sum_{j=1}^{\ell} \lambda_j \left( \sum_{i=1}^{n} p_{ji} x_i - r_j \right) + \sum_{k=1}^{m} \mu_k \left( \sum_{i=1}^{n} q_{ki} x_i - s_k \right). \tag{21}$$

*To obtain the sensitivities of the optimal value of the objective function to* $r_t$, $s_t$, $c_t$, $p_{t\ell}$ *or* $q_{t\ell}$, *following Theorem 1, we simply obtain the partial derivatives of the Lagrangian function with respect to the corresponding parameter, that is,*

$$\begin{aligned} \frac{\partial z_P^*}{\partial r_t} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s})}{\partial r_t} &= -\lambda_t^* & (22)\\ \frac{\partial z_P^*}{\partial s_t} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s})}{\partial s_t} &= -\mu_t^* & (23)\\ \frac{\partial z_P^*}{\partial c_t} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s})}{\partial c_t} &= x_t^* & (24)\\ \frac{\partial z_P^*}{\partial p_{t\ell}} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s})}{\partial p_{t\ell}} &= \lambda_t^* x_\ell^* & (25)\\ \frac{\partial z_P^*}{\partial q_{t\ell}} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}, \boldsymbol{r}, \boldsymbol{s})}{\partial q_{t\ell}} &= \mu_t^* x_\ell^*. & (26) \end{aligned}$$

**Example 4 (Dependence on a common parameter)** *Consider also the more complex case of all parameters depending on a common parameter $a$, i.e., the problem*

$$\text{Minimize } z_P = \sum_{i=1}^{n} c_i(a)x_i \tag{27}$$

*subject to*

$$\sum_{i=1}^{n} p_{ji}(a)x_i = r_j(a) \ : \ \lambda_j; \ \ j = 1, 2, \dots, \ell \tag{28}$$

$$\sum_{i=1}^{n} q_{ki}(a)x_i \leq s_k(a) \ : \ \mu_k; \ \ k = 1, 2, \dots, m. \tag{29}$$

*Then, the sensitivity of the optimal value of the objective function with respect to $a$ is given by (see Equation (21)):*

$$\begin{aligned}\frac{\partial z_P^*}{\partial a} = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)}{\partial a} &= \sum_{i=1}^{n} \frac{dc_i(a)}{da} x_i + \sum_{j=1}^{\ell} \lambda_j \left( \sum_{i=1}^{n} \frac{dp_{ji}(a)}{da} x_i - \frac{dr_j(a)}{da} \right) \\ &\quad + \sum_{k=1}^{m} \mu_k \left( \sum_{i=1}^{n} \frac{dq_{ki}(a)}{da} x_i - \frac{ds_k(a)}{da} \right)\end{aligned} \tag{30}$$

*Note that the cases in (22) to (26) are particular cases of (30).*

# 4 A General method for obtaining all sensitivities

The method developed in the previous section was limited to determining the sensitivities of the objective function. In what follows, and in order to simplify the mathematical derivations, the parameter vectors $\boldsymbol{b}$ and $\boldsymbol{c}$, used in (1)-(3) and (7)-(11), are assumed to be subsumed by $\boldsymbol{a}$.

In this section we present a powerful method that allows us determining all sensitivities at once, that is, the sensitivities of the optimal solutions $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, z^*)$ of the problems (1)-(3) and (4)-(5) to local changes in the parameters $\boldsymbol{a}$. To this end, we perturb or modify $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ and $z$ in such a way that the KKTC still hold. Let $J$ be the set of indices $j$ for which $g_j(\boldsymbol{x}^*, \boldsymbol{a}) = c_j$, and $m_j$ its cardinality. Thus, to obtain the sensitivity equations we differentiate (1) and the KKT conditions (7)-(11), as follows:

$$(\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{x} + (\nabla_{\boldsymbol{a}} f(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{a} - dz = 0 \tag{31}$$

$$\begin{aligned}&\left( \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \right) d\boldsymbol{x} \\ &+ \left( \nabla_{\boldsymbol{xa}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xa}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m} \mu_j^* \nabla_{\boldsymbol{xa}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \right) d\boldsymbol{a} \\ &+ \nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}) d\boldsymbol{\lambda} + \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) d\boldsymbol{\mu} = \boldsymbol{0}_n\end{aligned} \tag{32}$$

$$(\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{x} + (\nabla_{\boldsymbol{a}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{a} = \boldsymbol{0}_\ell \tag{33}$$

$$(\nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{x} + (\nabla_{\boldsymbol{a}} g_j(\boldsymbol{x}^*, \boldsymbol{a}))^T d\boldsymbol{a} = 0; \text{ if } \mu_j^* \neq 0; j \in J \tag{34}$$

where all the matrices are evaluated at the optimal solution, and redundant constraints have been removed.

It should be noted that the derivation above is based on results reported in Castillo et al. (2005a). It should also be noted that once an optimal solution of the estimation problem is known, binding inequality constraints are considered equality constraints and non-binding ones are disregarded. Note that this is appropriate as our analysis is just local. We assume local convexity around an optimal solution, which might not imply a globally optimal solution.

In matrix form, the system (31)-(34) can be written as:

$$\boldsymbol{M}\delta\boldsymbol{p} \quad = \quad \left[\begin{array}{c|c|c|c|c} \boldsymbol{F_x} & \boldsymbol{F_a} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{F_{xa}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{H_a} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x} & \boldsymbol{G_a} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right] \left[\begin{array}{c} d\boldsymbol{x} \\ \hline d\boldsymbol{a} \\ \hline d\boldsymbol{\lambda} \\ \hline d\boldsymbol{\mu} \\ \hline dz \end{array}\right] = \boldsymbol{0} \tag{35}$$

where the submatrices are defined below (corresponding dimensions in parentheses)

$$\boldsymbol{F_x}_{(1\times n)} \quad = \quad (\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{a}))^T \tag{36}$$

$$\boldsymbol{F_a}_{(1\times p)} \quad = \quad (\nabla_{\boldsymbol{a}} f(\boldsymbol{x}^*, \boldsymbol{a}))^T \tag{37}$$

$$\boldsymbol{F_{xx}}_{(n\times n)} \quad = \quad \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \tag{38}$$

$$\boldsymbol{F_{xa}}_{(n\times p)} \quad = \quad \nabla_{\boldsymbol{xa}} f(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xa}} h_k(\boldsymbol{x}^*, \boldsymbol{a}) + \sum_{j=1}^{m_J} \mu_j^* \nabla_{\boldsymbol{xa}} g_j(\boldsymbol{x}^*, \boldsymbol{a}) \tag{39}$$

$$\boldsymbol{H_x}_{(\ell\times n)} \quad = \quad (\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}))^T \tag{40}$$

$$\boldsymbol{H_a}_{(\ell\times p)} \quad = \quad (\nabla_{\boldsymbol{a}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}))^T \tag{41}$$

$$\boldsymbol{G_x}_{(m_J\times n)} \quad = \quad (\nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}))^T \tag{42}$$

$$\boldsymbol{G_a}_{(m_J\times p)} \quad = \quad (\nabla_{\boldsymbol{a}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}))^T . \tag{43}$$

The dimensions of all the above matrices are given in Table 1.

Table 1: Main matrices and their respective dimensions.

| | | | | |
|---|---|---|---|---|
| $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{a}} f(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*, \boldsymbol{a})$ |
| $n \times 1$ | $p \times 1$ | $n \times n$ | $n \times n$ | $n \times n$ |
| $\nabla_{\boldsymbol{xa}} f(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{xa}} h_k(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{xa}} g_j(\boldsymbol{x}^*, \boldsymbol{a}))$ | $\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{a}} \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a})$ |
| $n \times p$ | $n \times p$ | $n \times p$ | $n \times \ell$ | $p \times l$ |
| $\nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{a}} \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*, \boldsymbol{a})$ | $\nabla_{\boldsymbol{a}} g_j(\boldsymbol{x}^*, \boldsymbol{a})$ | $\boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a})$ |
| $n \times m_J$ | $p \times m_J$ | $n \times 1$ | $m_J \times 1$ | $m_J \times 1$ |
| $d\boldsymbol{x}$ | $d\boldsymbol{a}$ | $d\boldsymbol{\lambda}$ | $d\boldsymbol{\mu}$ | $dz$ |
| $n \times 1$ | $p \times 1$ | $\ell \times 1$ | $m_J \times 1$ | $1 \times 1$ |

Condition (35) can be written as

$$\boldsymbol{U} \left[\begin{array}{c} d\boldsymbol{x} \\ \hline d\boldsymbol{\lambda} \\ \hline d\boldsymbol{\mu} \\ \hline dz \end{array}\right] \quad = \quad \boldsymbol{S} d\boldsymbol{a} \tag{44}$$

where the matrices $\boldsymbol{U}$ and $\boldsymbol{S}$ are:

$$\boldsymbol{U} \quad = \quad \left[\begin{array}{c|c|c|c} \boldsymbol{F_x} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x^1} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right], \quad \boldsymbol{S} = -\left[\begin{array}{c} \boldsymbol{F_a} \\ \hline \boldsymbol{F_{xa}} \\ \hline \boldsymbol{H_a} \\ \hline \boldsymbol{G_a^1} \end{array}\right], \tag{45}$$

If the solution $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$ and $z^*$ is a non-degenerate regular point, then the matrix $\boldsymbol{U}$ is invertible and the solution of the system (44) is unique and it becomes

$$\left[\begin{array}{c} d\boldsymbol{x} \\ \hline d\boldsymbol{\lambda} \\ \hline d\boldsymbol{\mu} \\ \hline dz \end{array}\right] \quad = \quad \boldsymbol{U}^{-1}\boldsymbol{S}\; d\boldsymbol{a}. \tag{46}$$

Several partial derivatives can be simultaneously obtained if the vector $d\boldsymbol{a}$ in (46) is replaced by a matrix including several vectors (columns) with the corresponding unit directions. In particular, replacing $d\boldsymbol{a}$ by the unit matrix $\boldsymbol{I}_p$ in (46) all the partial derivatives are obtained, and the matrix with all partial derivatives is:

$$\left[\begin{array}{c} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{a}} \\ \dfrac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{a}} \\ \dfrac{\partial z}{\partial \boldsymbol{a}} \end{array}\right] \quad = \quad \boldsymbol{U}^{-1}\boldsymbol{S}. \tag{47}$$

For any vector $d\boldsymbol{a}$ the derivatives in both directions $d\boldsymbol{a}$ and $-d\boldsymbol{a}$ are obtained simultaneously.

Note that system (44) can be solved efficiently using a LU decomposition of matrix $\boldsymbol{U}$ and then applying forward and backward substitution with every column vector of matrix $\boldsymbol{S}$.

**Example 5 (Linear programming)** *Consider the following LP problem:*

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad z = \mathbf{c}^T\boldsymbol{x} \tag{48}$$

*subject to*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \;:\; \boldsymbol{\lambda}, \tag{49}$$

*where* $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, $\boldsymbol{b} = (b_1, b_2, \dots, b_m) \geq \boldsymbol{0}$, $\boldsymbol{A}$ *is a matrix of dimensions* $m \times n$ *with elements* $a_{ij}; i = 1, 2, \dots m; j = 1, 2, \dots, n$, *and* $\boldsymbol{\lambda}$ *are the dual variables.*

*Then, the local sensitivities become:*

$$\begin{array}{rclrclrcl} \dfrac{\partial z}{\partial c_j} & = & x_j; & \dfrac{\partial z}{\partial a_{ij}} & = & \lambda_i x_j; & \dfrac{\partial z}{\partial b_i} & = & -\lambda_i, \\[2ex] \dfrac{\partial x_j}{\partial c_k} & = & 0; & \dfrac{\partial x_j}{\partial a_{ik}} & = & -a^{ji}x_k & \dfrac{\partial x_j}{\partial b_i} & = & a^{ji}, \\[2ex] \dfrac{\partial \lambda_i}{\partial c_j} & = & -a^{ji}; & \dfrac{\partial \lambda_i}{\partial a_{\ell j}} & = & -a^{ji}\lambda_\ell; & \dfrac{\partial \lambda_i}{\partial b_\ell} & = & 0, \end{array} \tag{50}$$

*where* $a^{ji}$ *are the elements of* $\boldsymbol{A}^{-1}$.

*Observe that the equations (50) provide the sensitivities of the objective function, the primal variables and the dual variables with respect to all parameters of the general linear programming problem (48)-(49).*

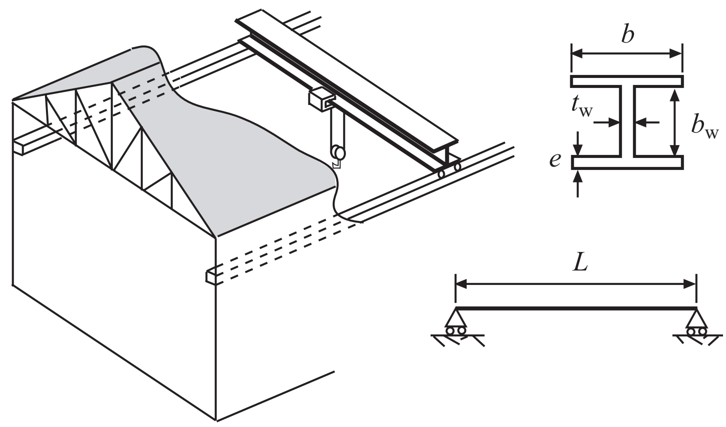


Figure 1: Graphical illustration of the bridge crane design and its cross section.

# 5 Examples of applications

## 5.1 Bridge crane design

In this Section, we apply an optimization method (minimize the cost) for designing an overhead crane (see Figure 1). This example appears in more detail in Castillo et al. (2003) and Conejo et al. (2006). In particular, we calculate the bridge girder dimensions that allow trolley travelling horizontally. It should be a box section fabricated from plate of structural steel, for the web, top plate and bottom plate, so as to provide for maximum strength at minimum dead weight. Maximum allowable vertical girder deflection should be a function of span.

Consider the girder and the cross section shown in Figure 1, where $L$ is the span or distance from centerline to centerline of runway rails, $b$ and $e$ are the flange width and thickness, respectively and $h_w$ and $t_w$ are the web height and thickness, respectively.

The set of variables involved in this problem can be partitioned into four subsets:

$\boldsymbol{d}$: *Optimization design variables.* They are the design variables which values are to be chosen by the optimization program to optimize the objective function (minimize the cost). In this crane example these variables are (see Figure 1):

$$\boldsymbol{d} = \{b, e, t_w, b_w\}.$$

$\boldsymbol{\eta}$: *Non-optimization design variables.* They are the set of variables which mean or characteristic values are fixed by the engineer or the code and must be given as data to the optimization program. In this bridge girder example:

$$\boldsymbol{\eta} = \{f_y, P, L, \gamma_y, E, \nu, c_y\}$$

where $f_y$ is the elastic limit of structural steel, $P$ is the maximum load supported by the girder, $L$ is the length of the span, $\gamma_y$ is the steel unit weight, $E$ is the Young modulus of the steel, $\nu$ is the Poisson modulus and $c_y$ is the steel cost.

$\boldsymbol{\kappa}$: *Random model parameters.* They are the set of parameters defining the random variability and dependence structure of the variables involved (see Castillo et al. (2003)). In this example:

$$\boldsymbol{\kappa} = \{\sigma_{f_y}, \sigma_P, \sigma_L, \sigma_{\gamma_y}\},$$

where $\sigma$ refers to the standard deviation of the corresponding variable.

$\boldsymbol{\psi}$: *Auxiliary or non-basic variables.* They are auxiliary variables which values can be obtained from the basic variables $\boldsymbol{d}$ and $\boldsymbol{\eta}$, using some formulas. In this example:

$$\boldsymbol{\psi} = \{W, I_{xx}, I_{yy}, I_t, G, \sigma, \tau, M_{cr}, \delta\},$$

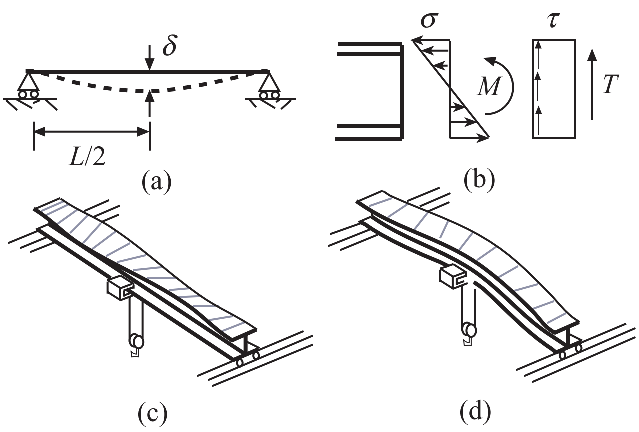


Figure 2: Illustration of the bridge girder modes of failure: (a) Maximum allowed deflection, (b) damage limit state of the steel upper and lower flanges and the steel web, (c) local buckling and (d) global buckling.

where $W$ is the girder bridge weight per unit length, $I_{\mathrm{xx}}, I_{\mathrm{yy}}$ are the moments of inertia, $I_{\mathrm{t}}$ is the torsional moment of inertia, $G = E/(2(1+\nu))$, $\sigma = M(h_w + e)/(2I_{xx})$, $\tau = T/(h_w t_w)$, $M_{\mathrm{cr}}$ is the critical moment for global buckling, and $\delta$ is the deflection at the center of the beam.

In the classical approach the safety factors are used as constraints and the variables are assumed to be deterministic, that is, either the mean or characteristic (extreme percentiles) values of the variables are used.

Assume that the following four failure modes are considered (see Figure 2):

1. *Maximum allowed deflection.* The maximum deflection safety factor $F_{\mathrm{d}}$ is defined (see Figure 2(a)) as

$$F_{\mathrm{d}} = \frac{\delta_{\max}}{\delta}, \tag{51}$$

where $\delta$ is the maximum deflection on the center of the girder, and $\delta_{\max}$ is the maximum deflection allowed by codes.

2. *Damage limit state of the steel upper and lower flanges.* We consider the ratio of the actual strength to actual stresses (see Figure 2(b)) as

$$F_{\mathrm{u}} = \frac{f_{\mathrm{y}}}{\sqrt{\sigma^2 + 3\tau^2}}, \tag{52}$$

where $F_{\mathrm{u}}$ is the corresponding safety factor, and $\sigma$ and $\tau$ are the normal and tangential stresses at the center of the beam, respectively.

3. *Damage limit state of the steel web.* The bearing capacity safety factor $F_{\mathrm{w}}$ is the ratio of the shear strength capacity to actual shear stress (see Figure 2(b)) at the center of the beam

$$F_{\mathrm{w}} = \frac{f_{\mathrm{y}}}{\sqrt{3}\tau}. \tag{53}$$

4. *Global Buckling.* The global buckling safety factor $F_{\mathrm{b}}$ is the ratio of the critical moment against buckling $M_{\mathrm{cr}}$ (see Figure 2(d)) of the cross section to the maximum moment applied at the center of the beam $M$

$$F_{\mathrm{b}} = \frac{M_{\mathrm{cr}}}{M}. \tag{54}$$

The gilder bridge is safe if and only if $F_{\mathrm{d}}, F_{\mathrm{u}}, F_{\mathrm{w}}$ and $F_{\mathrm{b}} \geq 1$.

To design the bridge crane its cost is minimized subject to the following constraints for the safety factors and reliability indices:

$$F_{\mathrm{d}}^0 = 1.15, \quad F_{\mathrm{u}}^0 = 1.5, F_{\mathrm{w}}^0 = 1.5, \quad F_{\mathrm{b}}^0 = 1.3,$$

$$\beta_{\mathrm{d}}^0 = 1.5, \quad \beta_{\mathrm{u}}^0 = 3.7, \beta_{\mathrm{w}}^0 = 3.7, \quad \beta_{\mathrm{b}}^0 = 3.2.$$

Table 2 gives the resulting sensitivities. Note that a very valuable information can be obtained from this table, for example:

1. The sensitivities of the reliability indexes with respect the mean values of the vertical load, $\mu_P$, and girder span length, $\mu_L$, are negative in all cases indicating the increasing probability of failure for every failure mode as it should be expected. For this reason the sensitivity of the cost with respect these variables is positive, because in order to keep the same safety level with an increase in the vertical load and the span length more money must be invested.

2. Analogously, sensitivities of the cost and reliability indexes are positive and negative, respectively, showing that an increase in the uncertainties ($\sigma_P$ and $\sigma_L$) produce an increase in the cost and a decrease on the reliability.

3. An increase of 1$ in the cost of the steel produces an increase in the total cost of the structure of 9842.876$ (see the corresponding entry in Table 2).

4. Regarding the sensitivities of the cost with respect to the lower bounds on global safety factors and reliability indexes it should be noted that only reliability indexes bounds ($\beta_d$ and $\beta_b$) related to maximum deflection ($d$) and global buckling ($b$) are active. This means that these failure modes are dominant so that if these constraints are enforced the remainder safety constraints are fulfilled. Note that sensitivity values are positive indicating that an increase on the safety requirements (decrease on the probabilities of failure) implies an increase in the cost. Note as well that from the economical point of view it is more important the global buckling ($b$) than the maximum deflection because the corresponding sensitivity is higher (77.858 versus 37.611).

## 5.2 The breakwater example

This example has been published by Castillo et al. (2006a). Consider the breakwater in Figure 3, where the main parameters are shown. Notice that these parameters define geometrically the different elements of the cross section and must be defined in the construction drawings. Our goal is an optimal design based on minimizing the initial construction costs subject to bounded yearly failure rates due to the 7 failure modes shown in Figure 4.

In Table 3 the sensitivities of the the yearly failure rates $r_m$ with respect to the design variables ($\bar{\boldsymbol{d}}$) for the optimal design are shown.

Note, for example, that for overtopping failure the only sensitivities different from zero are the ones related to dimensions defining the height of the caisson crest $h_o$, $h_b$, $h_s$, and $h_n$ and all are negative. This is obvious because when increasing any of these dimensions the composite breakwater height increases and, consequently, the probability of overtopping decreases. On the other hand, for the main armor layer failure, the only sensitivities different from zero are those related to the armor layer thickness $e$ (closely related to the weight of the pieces), the berm width $B_m$, and the height of the core $h_n$. The first value is negative because increasing the thickness implies increasing the weight of the rubble and therefore the probability of extracting pieces from the main layer decreases. The second value is positive, this means than increasing the berm width increases the probability of extracting pieces. Note that this result is not so obvious, this proves that sensitivity analysis provides a better understanding of how the models used work. And finally, the third sensitivity is positive, i.e., increasing the height of the core the probability of extracting pieces increases because it raises the berm level and there in less water depth protection against wave breaking.

Table 2: Sensitivities for the bridge crane design problem.

| $x$ | $\partial\text{cost}/\partial x$ | $\partial\beta_\text{u}/\partial x$ | $\partial\beta_\text{w}/\partial x$ | $\partial\beta_\text{b}/\partial x$ | $\partial\beta_d/\partial x$ |
|---|---|---|---|---|---|
| $b$ | $--$ | 12.851 | 0.000 | 46.864 | 17.717 |
| $e$ | $--$ | 280.902 | 0.000 | 993.835 | 408.587 |
| $t_\text{w}$ | $--$ | 352.458 | 698.939 | 122.267 | 108.587 |
| $h_\text{w}$ | $--$ | 11.974 | 7.687 | 0.448 | 23.088 |
| $\mu_P$ | 1.268 | $-0.008$ | $-0.005$ | $-0.011$ | $-0.011$ |
| $\mu_L$ | 746.662 | $-0.975$ | 0.000 | $-3.218$ | $-2.722$ |
| $\mu_{\gamma_\text{y}}$ | 30.125 | 0.000 | 0.000 | 0.000 | $-0.001$ |
| $\sigma_P$ | 3.312 | $-0.036$ | $-0.027$ | $-0.035$ | $-0.016$ |
| $\sigma_L$ | 149.935 | $-0.303$ | 0.000 | $-1.657$ | $-0.556$ |
| $\nu$ | 290.378 | 0.000 | 0.000 | $-3.730$ | 0.000 |
| $c_\text{y}$ | 9842.876 | 0.000 | 0.000 | 0.000 | 0.000 |
| $F_\text{u}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $F_\text{w}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $F_\text{b}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $F_\text{d}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $\beta_\text{u}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $\beta_\text{w}$ | 0.000 | $--$ | $--$ | $--$ | $--$ |
| $\beta_\text{b}$ | 77.858 | $--$ | $--$ | $--$ | $--$ |
| $\beta_\text{d}$ | 37.611 | $--$ | $--$ | $--$ | $--$ |

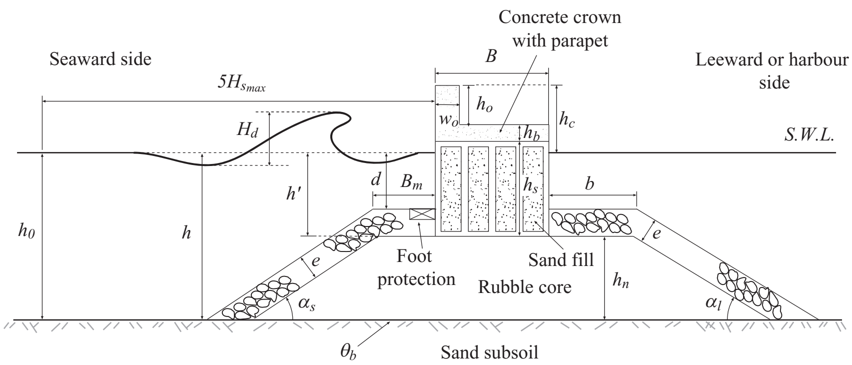


Figure 3: Composite breakwater showing the geometric design variables.

| $\bar{d}_i$ | $\frac{\partial r_s}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_b}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_c}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_d}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_{rs}}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_o}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) | $\frac{\partial r_a}{\partial \bar{d}_i}\lvert\bar{d}_i\rvert$ (\$) |
|---|---|---|---|---|---|---|---|
| $b$ | - | - | -0.000192 | -0.000107 | -0.001284 | - | - |
| $B$ | -0.004348 | -0.008655 | -0.001718 | -0.002127 | -0.005488 | - | - |
| $B_m$ | - | - | - | - | - | - | 0.018256 |
| $e$ | 0.000365 | 0.000472 | -0.000115 | 0.000064 | -0.000965 | - | -0.022008 |
| $h_b$ | -0.001119 | -0.000713 | -0.000182 | -0.000408 | 0.000279 | -0.012560 | - |
| $h_n$ | -0.000180 | -0.000023 | 0.000550 | -0.000477 | -0.002465 | -0.017730 | 0.014502 |
| $h_o$ | 0.000311 | 0.000483 | 0.000105 | 0.000176 | 0.000516 | -0.010712 | - |
| $h_s$ | -0.004037 | -0.002512 | -0.000644 | -0.001464 | 0.001110 | -0.047406 | - |
| $\alpha_\ell$ | - | - | 0.000840 | 0.000245 | 0.004256 | - | - |
| $\alpha_s$ | - | - | - | - | - | - | - |

Table 3: Breakwater example. Sensitivities of the the yearly failure rates $r_m$ with respect to the design variables ($\bar{\boldsymbol{d}}$).

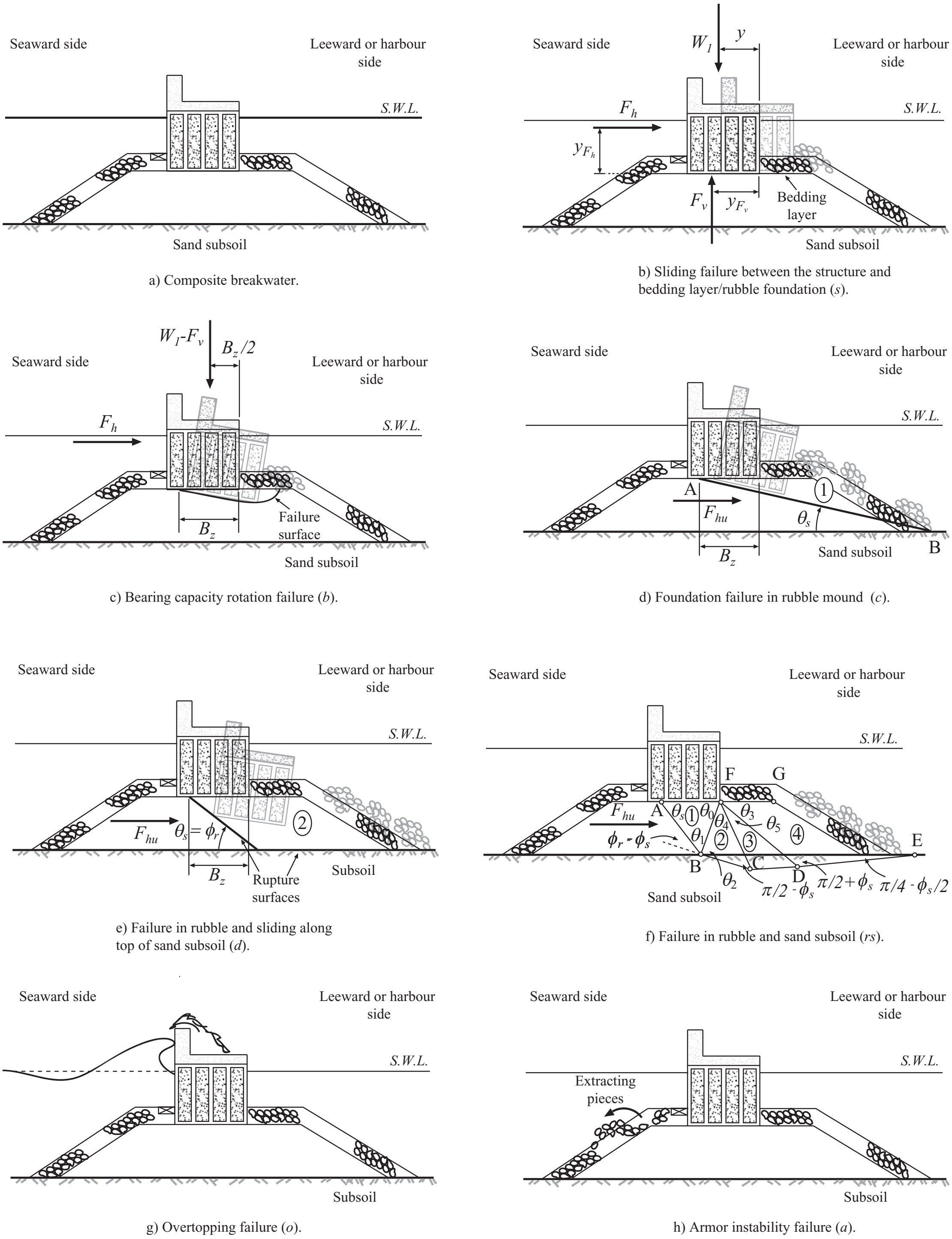


Figure 4: Section of the composite breakwater and the seven failure modes considered in the example.

## 5.3 The retaining wall example

A retaining wall problem involves many variables, such as all wall dimensions, backfill slope, concrete and steel strength, allowable soil bearing pressure, backfill properties, etc. For example, consider the

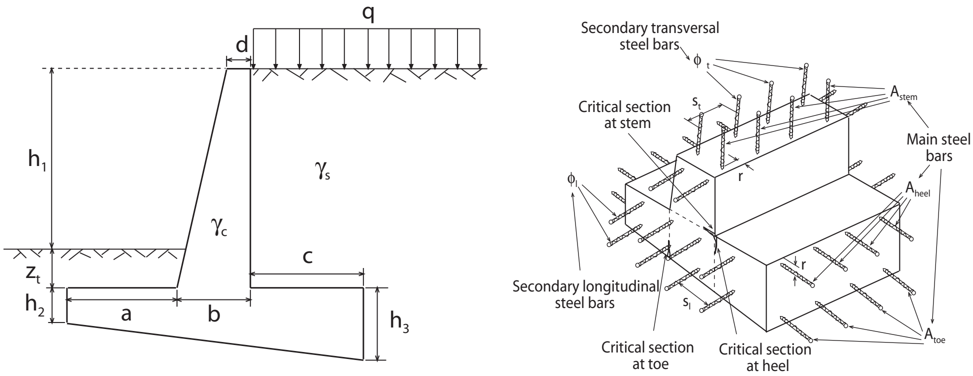


Figure 5: Geometrical description of the wall and details of the bar reinforcement.

retaining wall in Castillo et al. (2004) as shown in Figure 5 defining the geometry of the wall (note that the wall is defined in parametric form) and the reinforcement.

Assume that the following failure modes are considered: Sliding, overturning, bearing capacity, and stem, toe and heel flexural and shear failures.

Since, including the three flexural and three shear, we have 9 different failure modes, we define the set $I_f$ of failure modes as

$$I_f = \{s, t, b, stem, toe, heel, sstem, stoe, sheel\}$$

The analysis of these failures can be performed by considering

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) = F_i; \;\; i \in I_f \tag{55}$$

where $g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}})$ is the ratio of two opposing magnitudes, such as stabilizing to overturning forces, strengths to ultimate stresses, etc., and the stability coefficients, $F_i$, are random quantities. The wall will be safe if and only if

$$F_s, F_t, F_b, F_{stem}, F_{toe}, F_{heel}, F_{sstem}, F_{stoe}, F_{sheel} \geq 1.$$

The proposed method for the wall design consists of minimizing the total cost, that is,

$$\underset{\bar{\boldsymbol{d}}}{\text{Minimizing}} \quad h(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \tag{56}$$

subject to

$$g_i(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}) \geq F_i^0; \;\; i \in I_f, \tag{57}$$

and

$$\beta_{F_i}(\bar{\boldsymbol{d}}, \bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}) \geq \beta_i^0; \;\; i \in I_f. \tag{58}$$

where $F_i^0$ and $\beta_i^0$; $i \in I_f$ are the lower bounds on the global safety factors and reliability indexes, respectively.

The final result of the above problem is an optimal classical design ($\bar{\boldsymbol{d}}^*$) with the resulting safety factors, which is at the same time an optimal probability-based design, as it satisfies the probability requirements.

A sensitivity analysis can be easily performed using Theorem 1 so that sensitivities of the cost and reliability indexes with respect to data are easily obtained. Table 4 gives the cost sensitivities with respect to the data values. They indicate how much the objective function changes with a very small unit increment of the corresponding parameter. The following observations are pertinent:

Table 4: Retaining wall example. Cost sensitivities with respect to the data values in the wall illustrative example.

| | | | | | |
|---|---|---|---|---|---|
| $c_c$ | $c_t$ | $c_{st}$ | $c_{ex}$ | $r$ | $\phi_l$ |
| 11.511 | 12.811 | 2730.090 | 56.743 | 738.238 | 3.667 |
| $\phi_t$ | $s_l$ | $s_t$ | $F_t^0$ | $F_s^0$ | $F_b^0$ |
| 5.243 | -110.025 | -157.291 | 0.000 | 193.430 | 55.728 |
| $F_{stem}^0$ | $F_{toe}^0$ | $F_{heel}^0$ | $F_{sstem}^0$ | $F_{stoe}^0$ | $F_{sheel}^0$ |
| 27.349 | 8.287 | 24.936 | 1.427 | 159.455 | 130.364 |
| $\bar{\gamma}_c$ | $\bar{\gamma}_s$ | $\bar{\gamma}_{st}$ | $\bar{f}_c$ | $\bar{f}_y$ | $\bar{\sigma}_{soil}$ |
| -5.595 | 22.042 | 2.087 | -0.024 | -0.245 | -448.968 |
| $\bar{\tau}_{max}$ | $\bar{\mu}_{crit}$ | $\bar{q}$ | $\bar{k}_a$ | $\bar{k}_p$ | $\bar{h}_1$ |
| -1360.752 | -773.721 | 11.666 | 2492.194 | -18.392 | 490.227 |

1. The sensitivities of the cost with respect the material costs of concrete, timber, steel and excavation ($c_c$,$c_t$,$c_{st}$, and $c_{ex}$) are 11.511, 12.811, 2730.09 and 56.743, respectively. In order to know which one is more important relative sensitivities are calculated $|c_c|\partial \text{cost}/\partial c_c = 759.726$, $|c_t|\partial \text{cost}/\partial c_t = 153.7200$, $|c_{st}|\partial \text{cost}/\partial c_{st} = 163.8054$, and $|c_{ex}|\partial \text{cost}/\partial c_{ex} = 204.2748$, where it is shown that the concrete cost price is the most relevant price in the optimal solution.

2. Sensitivities of the cost with respect to the lower bounds of the global safety factors are also given. Note that all the values are positive indicating that an increase in the safety requirements implies an increase in the cost. Note that as the probability of failure due to overturning is much lower than the rest of the failure modes, the lower bound on its corresponding safety factor is inactive, and therefore the corresponding sensitivity is null.

3. Note also that the sensitivity with respect the mean value of the surcharge ($\bar{q}$) is equal to 11.666. This indicates an increase in the cost if the surcharge increases in order to maintain the same safety level.

# 6 The case of Calculus of Variations

Consider the case of a slope as that in Figure 6 which is in a critical situation (some cracks have appeared on top of the slope and sliding through the critical sliding curve, determined by calculus of variations and shown in the figure, is imminent). In cases similar to this, the engineer is asked to make quick decisions on how to improve the precarious stability of the slope. In these occasions it is extremely important to know where changes of the slope profile must be made and what soil properties must be improved to increase the slope safety. To this aim, sensitivity analysis is the best technique because it points out the main causes of instability and immediately suggests the most effective actions to make the adequate corrections. In our slope example, one can ask what is the sensitivity of the safety factor to soil properties and to the slope profile. Once this information is available, the engineer can immediately know where the slope profile must be modified and what properties of the soil are the most influential on the slope safety. Thus, the answers to questions as what changes in the slope profile produce the largest improvement of the safety factor? or what changes in the soil strength are the most effective to avoid instability? can be obtained via sensitivity analysis, and no other techniques give better and more precise answers.

As in this example, there are many practical problems in which the calculus of variations is the natural and more adequate mathematical model to be used and where sensitivity analysis becomes appropriate.

Since calculus of variations problems are optimization problems subject to constraints, and Theorem 1 and Expression (47) give the sensitivities of the objective function optimal value, and the primal and the dual solutions with respect to all the parameters in a neat and straightforward form for standard

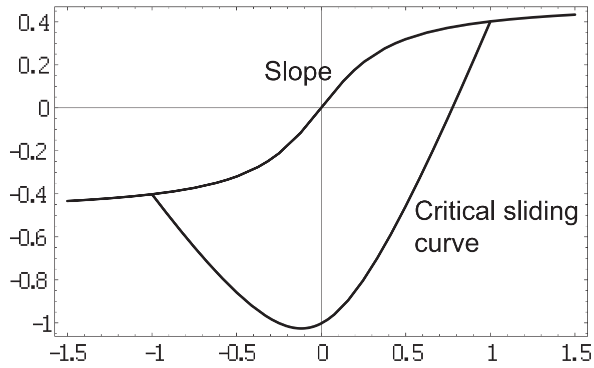


Figure 6: Critical sliding curve passing through the given end-points.

optimization problems, the following question can be asked: are there equivalent formulations for the calculus of variations problems?

The answer is positive and is given in this paper. In fact, parallel results exist for optimal control (see Malanowski and Maurer (1996)).

## 6.1 Some required background on calculus of variations

Let us consider the constrained classical problem of calculus of variations (see Krasnov et al. (1973); Elsgolc (1962); Goldstine (1980)):

$$\underset{u(t)}{\text{Minimize}} \quad J = J(u(t); a, b) = \int_a^b F(t, u(t), u'(t))\, dt \tag{59}$$

subject to

$$\mathcal{H}_i(u(t)) = \int_a^b H_i(t, u(t), u'(t))\, dt = 0, \quad i = 1, 2, \ldots, m. \tag{60}$$

Note that there are important differences between problem (1)-(3) (an optimization problem) and problem (59)-(60) (a calculus of variations problem). In the first one minimizes functions, and in the second, one minimizes functionals. In the first, one minimizes with respect to variables and in the second with respect to functions. Finally, optimization problems constraints involve functions, while calculus of variations constraint can involve functionals, as in (60).

For $u(t)$ to be an optimal solutions, it must satisfy the following conditions:

1. The *Euler-Lagrange equation*:

$$\begin{aligned} 0 \;=\; & \frac{\partial F}{\partial u}(t, u^*, (u^*)') - \frac{d}{dt}\left(\frac{\partial F}{\partial u'}(t, u^*, (u^*)')\right) \\ & + \boldsymbol{\lambda}^* \cdot \left(\frac{\partial \mathbf{H}}{\partial u}(t, u^*, (u^*)') - \frac{d}{dt}\left(\frac{\partial \mathbf{H}}{\partial u'}(t, u^*, (u^*)')\right)\right), \end{aligned} \tag{61}$$

   which introducing the following notation

$$\mathcal{E}_u(F(t, u^*, (u^*)')) = \frac{\partial F}{\partial u}(t, u^*, (u^*)') - \frac{d}{dt}\left(\frac{\partial F}{\partial u'}(t, u^*, (u^*)')\right),$$

   and

$$\mathcal{L}(t, u^*, (u^*)', \boldsymbol{\lambda}^*) = F(t, u^*, (u^*)') + \boldsymbol{\lambda}^* \cdot \mathbf{H}(t, u^*, (u^*)'),$$

   becomes

$$\mathcal{E}_u(\mathcal{L}(t, u^*, (u^*)', \boldsymbol{\lambda}^*)) = 0. \tag{62}$$

2. If $a, b, u(a)$ and $u(b)$ are fixed, we have *fixed end conditions* (see Figure 7(a)), i.e.,

$$u(a) = u_a; \quad u(b) = u_b. \tag{63}$$

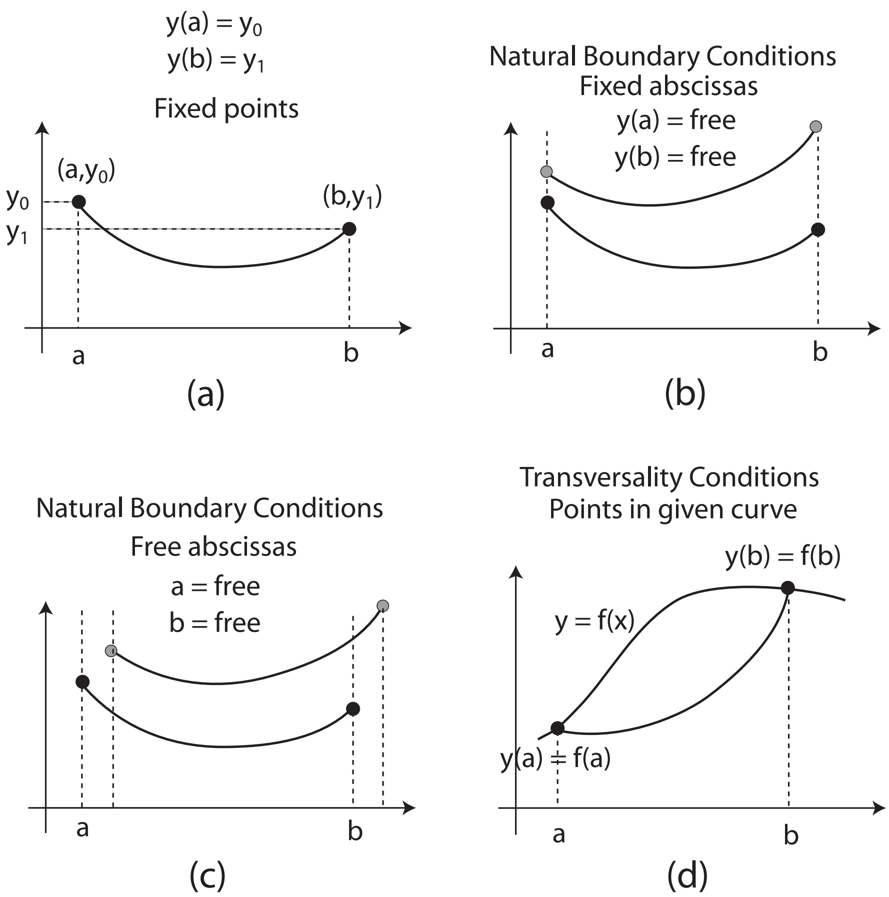


Figure 7: Illustration of several end conditions.

3. If some of the values $u(a)$ or $u(b)$ are not fixed, we have the *natural boundary condition* (see Figure 7(b)),

$$\frac{\partial \mathcal{L}}{\partial u'}(a, u^*(a), (u^*)'(a), \boldsymbol{\lambda}^*) = 0, \tag{64}$$

if $u(a)$ is free, or the corresponding equation if $u(b)$ is free.

4. Similarly, if the end-points $a$ or $b$ are free, we have the *natural boundary condition* (see Figure 7(c))

$$\mathcal{L}(a, u^*(a), (u^*)'(a), \boldsymbol{\lambda}^*) = 0, \tag{65}$$

if $a$ is free, or the corresponding equation if $b$ is free.

5. Finally, if the end-point $t = a$ is on a given curve $\psi(t)$, as in the slope stability example, we have the *transversality conditions* (see Figure 7(d))

$$\begin{aligned} \mathcal{L}(a, u^*(a), (u^*)'(a), \boldsymbol{\lambda}^*) + (\psi'(a) - (u^*)'(a)) \left( \frac{\partial \mathcal{L}}{\partial u'}(a, u^*(a), (u^*)'(a), \boldsymbol{\lambda}^*) \right) = 0, \\ u^*(a) = \psi(a), \end{aligned} \tag{66}$$

or the corresponding equations in case $t = b$ lies on the curve $\psi(t)$.

Equations (62) to (66) are the well known necessary conditions for an extremum in calculus of variations, which have been extensively used and applied to many practical problems.

Equation (62) together with the fixed end-point conditions or the corresponding natural or transversality conditions in (64) to (66), lead to a boundary value problem (BVP) which allow us solving the calculus of variations initial minimization problem.

## 6.2 Sensitivity analysis in calculus of variations

In this section we consider a parametric family of calculus of variations problems and we analyze how the corresponding optimal solutions change when parameters are modified. We can consider the case of finite parameters, and the case of data functions (infinitely many parameters).

### 6.2.1 Sensitivity analysis for a finite number of parameters

More precisely, consider the problem

$$\underset{u(t)}{\text{Minimize}} \quad J(u;\mathbf{p}) = \int_{a(\mathbf{p})}^{b(\mathbf{p})} F(t,u,u';\mathbf{p})\,dt \tag{67}$$

subject to

$$\mathcal{H}_i(u;\mathbf{p}) = \int_{a(\mathbf{p})}^{b(\mathbf{p})} H_i(t,u,u';\mathbf{p})\,dt = 0, \quad i=1,2,\ldots,m, \tag{68}$$

where $\mathbf{p}=(p_1,p_2,\ldots,p_k)\in\mathbb{R}^k$ is the vector of parameters.

In this case we have the following theorem, which is the parallel version of Theorem (1) for calculus of variations.

**Theorem 2** *The vector of sensitivities of the objective function of the primal problem (67)–(68) with respect to* $\mathbf{p}$ *is given by*

$$\begin{aligned}\frac{\partial J^*(u;\mathbf{p})}{\partial \mathbf{p}} &= \left(\int_{a(\mathbf{p})}^{b(\mathbf{p})} \frac{\partial \mathcal{L}}{\partial \mathbf{p}}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\mathbf{p})\,dt\right)\\ &+ \left.\mathcal{L}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\mathbf{p})\right|_{b(\mathbf{p})} b'(\mathbf{p}) - \left.\mathcal{L}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\mathbf{p})\right|_{a(\mathbf{p})} a'(\mathbf{p})\\ &+ \left.\frac{\partial \mathcal{L}}{\partial u'}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\mathbf{p})\right|_{b(\mathbf{p})} \delta u'(b(\mathbf{p}))b'(\mathbf{p}) - \left.\frac{\partial \mathcal{L}}{\partial u'}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\mathbf{p})\right|_{b(\mathbf{p})} \delta u'(a(\mathbf{p}))a'(\mathbf{p})\end{aligned} \tag{69}$$

*which is the gradient of its Lagrangian function with respect to* $\mathbf{p}$ *evaluated at the optimal solution* $u^*$, $\boldsymbol{\lambda}^*$. ∎

### 6.2.2 Sensitivity analysis for infinitely many parameters

A similar analysis as that performed in subsection 6.2.1 can be developed if the $k$-dimensional parameter $\mathbf{p}$ is changed by a vector data function $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_r):\mathbb{R}\to\mathbb{R}^r$.

More precisely, consider the problem

$$\underset{u(t)}{\text{Minimize}} \quad J(u;\boldsymbol{\phi}) = \int_{a}^{b} F(t,u,u';\boldsymbol{\phi},\boldsymbol{\phi}')\,dt \tag{70}$$

subject to

$$\mathcal{H}_i(u;\boldsymbol{\phi}) = \int_{a}^{b} H_i(t,u,u';\boldsymbol{\phi},\boldsymbol{\phi}')\,dt = 0, \quad i=1,2,\ldots,m, \tag{71}$$

In this case, the counterpart of Theorem 1 is the following theorem.

**Theorem 3** *The vector of sensitivities of the objective function of the primal problem (70)–(71) with respect to $\boldsymbol{\phi}(t)$ is given by*

$$\begin{aligned}\frac{\partial J^*(u;\boldsymbol{\phi})}{\partial \phi_i(t)} &= \mathcal{E}_{\phi_i}(\mathcal{L}(t,u^*,(u^*)',\boldsymbol{\lambda}^*;\boldsymbol{\phi})) \\ &+ \frac{\partial \mathcal{L}}{\partial u'}(t,u^*,(u^*)';\boldsymbol{\phi},\boldsymbol{\phi}')\delta u(t)\bigg|_a^b + \mathcal{L}(t,u^*,(u^*)';\boldsymbol{\phi})\delta(t)|_a^b \\ &+ \int_a^b \mathcal{E}_{\phi_i}(F(t,u^*,(u^*)';\boldsymbol{\phi},\boldsymbol{\phi}'))\delta\phi_i(t)\,dt \\ &+ \frac{\partial \mathcal{L}}{\partial \phi_i'}(t,u^*,(u^*)';\boldsymbol{\phi},\boldsymbol{\phi}')\cdot\delta\phi_i(t)\bigg|_a^b, \qquad (72)\end{aligned}$$

*which is the gradient of its Lagrangian function with respect to $\boldsymbol{\phi}$ evaluated at the optimal solution $u^*$, $\boldsymbol{\lambda}^*$.* ■

The practical consequences of Theorems 2 and 3 are that direct formulas for the sensitivities are available, while the remaining sensitivities (of primal and dual variables) are more difficult to obtain. In this case one needs to perturb the Euler-Lagrange equations (61) and the auxiliary conditions (63)-(66) (see a detailed derivation in Castillo et al. (2006b)).

## 6.3 A slope stability problem

In this section we present a slope stability problem which will appear in Castillo et al. (2006b). Slope stability analysis (see Castillo and Luceño (1982, 1983)) consists of determining the safety factor $F$ (the ratio of the resisting to sliding forces or moments) associated with the worst sliding line. Since each of these forces and moments can be given as a functional, the problem can be stated as the minimization of a quotient of two functionals.

Castillo and Revilla (1975, 1976); Castilllo and Revilla (1977), Revilla and Castillo (1977), and Luceño and Castillo (1980) based on the Janbu method (see Janbu (1957)) proposed for a purely cohesive soil the following functional:

$$Q = \frac{Q_N}{Q_D} = \frac{\displaystyle\int_{t_0}^{t_1} f_1(t,u(t),u'(t))dt}{\displaystyle\int_{t_0}^{t_1} f_2(t,u(t),u'(t))dt} = \frac{\displaystyle\int_{t_0}^{t_1}(1+u'^2(t))dt}{\displaystyle\int_{t_0}^{t_1}(\tilde{u}(t)-u(t))u'(t)dt}, \qquad (73)$$

where $f_1(t,u(t),u'(t))$ and $f_2(t,u(t),u'(t))$ and $Q_N$ and $Q_D$ are the subintegral functions and the functionals in the numerator and denominator, respectively, $Q = \dfrac{F}{V}$, $F$ is the safety factor, $V = \dfrac{c}{\gamma H}$, $c$ is the cohesion of the soil, $\gamma$ is the unit weight of the soil, $H$ is the slope height, $t_1$ and $t_2$ are the $t$-coordinates of the sliding line end-points, $\bar{u}(t)$ is the slope profile (ordinate at point $t$), $u(t)$ is the ordinate of the sliding line at point $t$, and (see Figure 6)

$$\tilde{u}(t) = \frac{1}{\pi}\arctan(\pi t); \quad -\infty < t < \infty. \qquad (74)$$

The $t$ and $u(t)$ values have been adequately normalized by dividing the true coordinates $x$ and $y(x)$, respectively, by the slope height $H$.

The Euler-Lagrange equation for this non-standard problem of calculus of variations is (see Euler (1744) and Revilla and Castillo (1977))

$$Q = \frac{f_{1u}(t,u(t),u'(t)) - \dfrac{d}{dt}f_{1u'}(t,u(t),u'(t))}{f_{2u}(t,u(t),u'(t)) - \dfrac{d}{dt}f_{2u'}(t,u(t),u'(t))} = \frac{2u''(t)}{\tilde{u}'(t)},$$

that is,

$$u''(x) = \frac{Q\tilde{u}'(t)}{2} \quad \Rightarrow \quad u'(t) = \frac{Q}{2}\tilde{u}(t) + B \quad \Rightarrow \quad u(t) = \frac{Q}{2}\int \tilde{u}(t)dt + Bt + C,$$

where $B$ and $C$ are arbitrary constants.

Then, Equation (74) provides the set of extremals

$$u(t) = \frac{Q}{2}\left[\frac{t\arctan(\pi t)}{\pi} - \frac{\log(1+\pi^2 t^2)}{2\pi^2}\right] + Bt + C.$$

In this case, the critical sliding line is infinitely deep (a well known result to soil mechanics experts). Thus, to simplify, we consider only the sliding lines passing through the end-points at $t_0 = -1$ and $t_1 = 1$. Then, the constants $B$, $C$ and $Q$ must satisfy the end-point conditions

$$\tilde{u}(t) = u(t); \quad t = t_0, t_1,$$

and Equation (73). Solving this system one gets

$$B = 0.401907, \quad C = -1.00284, \quad Q = 7.13684,$$

with a critical sliding line of equation

$$u(t) = -1.00284 + 0.401907\,t + 3.56842\left(\frac{t\ \arctan(\pi\, t)}{\pi} - \frac{\log(1+\pi^2\, t^2)}{2\,\pi^2}\right),$$

which is plotted in Figure 6.

The numerator $Q_N$ and denominator $Q_D$ of $Q$ in (73) become

$$Q_N = \int_{-1}^{1} (1 + (u')^2(t))dt = 4.64612$$

and

$$Q_D = \int_{-1}^{1} (\tilde{u}(t) - u(t))u'(t)dt = 0.651005.$$

To illustrate the proposed sensitivity method we consider also the parameterized problem

$$\underset{u(t)}{\text{Minimize}} \quad Q_N = \int_{-1}^{1} (1 + (u')^2(t))dt \tag{75}$$

subject to

$$Q_D = \int_{-1}^{1} \frac{1}{p}(\tilde{u}(t) - u(t))u'(t)dt = 1. \tag{76}$$

**Sensitivity analysis using the proposed methods**

The objective function sensitivity can be calculated using Theorem 2 as follows

$$\begin{aligned}
\frac{\partial Q_N^*}{\partial p} &= \int_{-1}^{1} \frac{\partial(1 + (u^*)'^2(t) + \dfrac{\lambda^*}{p}(\tilde{u}(t) - u^*(t))(u^*)'(t))}{\partial p} dt \\
&= -\frac{\lambda^*}{p^2}\int_{-1}^{1} (\tilde{u}(t) - u^*(t))(u^*)'(t)dt = -\frac{\lambda^*}{p} = 10.9628p.
\end{aligned}$$

and perturbing the necessary conditions and applying Theorem 3, one additionally gets

$$\frac{\partial \lambda^*}{\partial p} = -21.9256p \tag{77}$$

$$\frac{\partial u^*(t)}{\partial p} = -1.54045 + 5.4814\left(\frac{t\,\arctan(\pi\, t)}{\pi} - \frac{\log(1+\pi^2\, t^2)}{2\,\pi^2}\right), \tag{78}$$

$$\frac{\partial Q_N^*(u,\phi)}{\partial \tilde{u}(t)} = \mathcal{E}_{\tilde{u}}(\mathcal{L}(t,u^*(t),(u^*)'(t),\lambda^*;p)) = \frac{(2.323058+5.4814p^2)\tilde{u}'(t)}{p^2}, \tag{79}$$

which for $p = 1, d = 1$ is shown in Figure 8. As expected, an increment of soil on the right horizontal part of the slope reduces $Q$ (and then, the safety factor), and gives a negative sensitivity. Similarly, an increment of soil on the left horizontal part of the slope increases $Q$ (and then, the safety factor), and the sensitivity is positive. This immediately suggests the required actions to stabilize the slope.

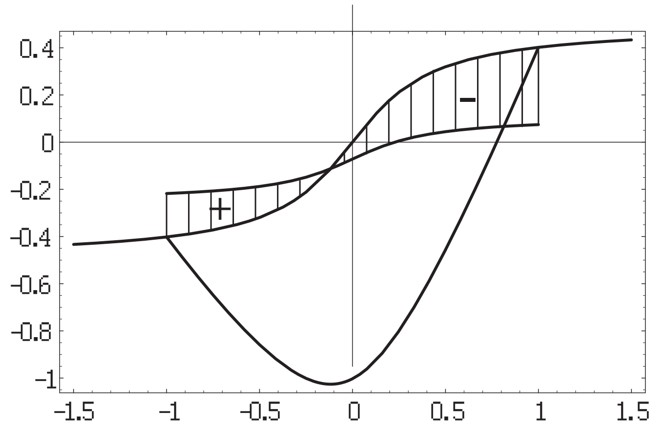


Figure 8: Sensitivity of $Q$ with respect to the slope profile.

To derive the sensitivity of the extremal sliding curve with respect to the slope profile, we use the indicated perturbation techniques with Dirac function perturbations, to get the results shown in Figure 9 (see Castillo et al. (2006b)).

# 7 Conclusions

The main conclusions from this paper are:

1. There exist very simple and closed formulas for sensitivity analysis of the objective function and the primal and dual variables with respect to data in regular cases of linear programming problems.

2. In the case of regular cases of non-linear programming a methodology has been given to perform a sensitivity analysis of the objective function and the primal and dual variables with respect to data. This implies solving the optimization problem and then constructing and solving a linear system of equations.

3. In non-regular cases a more involved methodology is required, which is given in another paper.

4. Sensitivity analysis can be done for calculus of variations problems in a similar way as it is done in optimization problems (linear and non-linear) and in optimal control problems, obtaining results for calculus of variations that are the parallel versions of those for the other problems.

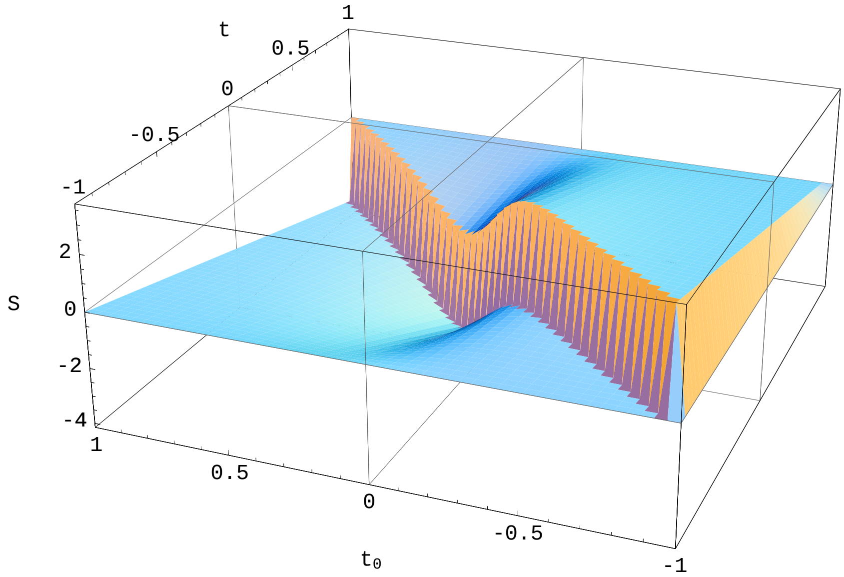


Figure 9: Sensitivity of the sliding curve with respect to the slope profile.

5. Theorems 2 and 3 are the counterparts of Theorem 1 for calculus of variations in the finite and infinite cases, respectively. They allow obtaining closed formulas for the objective function sensitivities with respect to the data.

6. For calculus of variations, the sensitivities of the primal and dual variables and functions with respect to variable and function type data require perturbation analysis, as indicated in the paper.

7. The practical applications presented in this paper have illustrated and clarified the theory, and demonstrated the goodness of the proposed technique, together with the importance of the practical applications that can benefit from the proposed methods.

# References


Bazaraa, M. S., Sherali, H. D., and Shetty, C. M. (1993). *Nonlinear Programming, Theory and Algorithms.* John Wiley & Sons, New York, second edition.

Castilllo, E. and Revilla, J. (1977). One application of the calculus of variations to the stability of slopes. In *Proceedings of the 9th International Conference on Soil Mechanics and Foundations Engineering*, volume 2, pages 25–30, Tokyo.

Castillo, C., Mínguez, R., Castillo, E., and Losada, M. (2006a). Engineering design method with failure rate constraints and sensitivity analysis. example application to composite breakwaters. *Coastal Engineering*, pages 1–25.

Castillo, E., Conejo, A., and Aranda, E. (2006b). Sensitivity analysis in calculus of variations. some applications. *Siam Review.* submitted.

Castillo, E., Conejo, A., Castillo, C., Mínguez, R., and Ortigosa, D. (2005a). A perturbation approach to sensitivity analysis in nonlinear programming. *Journal of Optimization Theory and Applications*, 127(3).

Castillo, E., Conejo, A., Mínguez, R., and Castillo, C. (2003). An alternative approach for addressing the failure probability-safety factor method with sensitivity analysis. *Reliability Engineering and System Safety*, 82:207–216.

Castillo, E., Conejo, A., Mínguez, R., and Castillo, C. (2005b). A closed formula for local sensitivity analysis in mathematical programming. *Engineering Optimization*. In press.

Castillo, E., Conejo, A., Pedregal, P., García, R., and Alguacil, N. (2001). *Building and Solving Mathematical Programming Models in Engineering and Science*. John Wiley & Sons Inc., New York. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts.

Castillo, E., Gutiérrez, J. M., and Hadi, A. S. (1996). Sensitivity analysis in discrete Bayesian networks. *IEEE Transactions on Systems, Man and Cybernetics*, 26:412–423.

Castillo, E. and Luceño, A. (1982). A critical analysis of some variational methods in slope stability analysis. *International Journal for Numerical and Analytical Methods in Geomechanics*, 6:195–209.

Castillo, E. and Luceño, A. (1983). Variational methods and the upper bound theorem. *Journal of Engineering Mechanics (ASCE)*, 109(5):1157–1174.

Castillo, E., Mínguez, R., Ruíz-Terán, A., and Fernández-Canteli, A. (2004). Design and sensitivity analysis using the probability-safety-factor method. An application to retaining walls. *Structural Safety*, 26:159–179.

Castillo, E. and Revilla, J. (1975). El cálculo de variaciones y la estabilidad de taludes. *Boletín del Laboratorio del Transporte y Mecánica del Suelo José Luis Escario*, 108:31–37.

Castillo, E. and Revilla, J. (1976). Una aplicación del cálculo de variaciones a la estabilidad de taludes. *Revista del Laboratorio del Transporte y Mecánica del Suelo José Luis Escario*, 115:3–23.

Conejo, A., Castillo, E., R., M., and García-Bertrand, R. (2006). *Decomposition Techniques in Mathematical Programming. Engineering and Science Applications*. Springer, Berlin, Heildelberg.

Elsgolc, L. (1962). *Calculus of Variations*. Pergamon Press, London-Paris-Frankfurt.

Euler, L. (1744). *"Metod Nakhozhdeniia Krivykh Linii, Obladiaiushchikh Svoistvami Maksimuma Libo Minimuma, Ili Reshenie Izoperimetricheskoi Zadachi Vziatoi v Samom Shirokom Smysle" (Method of Finding Curves Possessing Maximum or Minimum Properties, or The Solution of the Isoperimetric Problem Taken in Its Broadest Sense)*. GITTL. Translated from the 1744 ed.

Goldstine, H. H. (1980). *A history of the calculus of variations from the 17th to the 19th Century*. Springer Verlag, New York, Heildelberg, Berlin.

Janbu, N. (1957). Earth pressure and bearing capacity calculations by generalized procedure of slices. In *Proceedings of the 4th International Conference on Soil Mechanics and Foundations Engineering*, London.

Krasnov, M. A., Makarenko, G. I., and Kiselev, A. I. (1973). *Calculus of Variations*. Nauka, Moscow.

Luceño, A. and Castillo, E. (1980). Análisis crítico de los métodos variacionales aplicados a la estabilidad de taludes. *Boletín del Laboratorio del Transporte y Mecánica del Suelo José Luis Escario*, 45:3–14.

Luenberger, D. G. (1984). *Linear and Nonlinear Programming*. Addison-Wesley, Reading, Massachusetts, second edition.

Malanowski, K. and Maurer, H. (1996). Sensitivity analysis for parameteric optimal control problems with control-state constraints. *Computational Optimization and Applications*, 5:253–283.

Revilla, J. and Castillo, E. (1977). The calculus of variations applied to stability of slopes. *Geotechnique*, 27(1):1–11.

# Contents