# An Optimal Engineering Design Method That Combines Safety Factors and Failure Probabilities. Application to rubble-mound Breakwaters

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**Abstract**

The paper presents a new method for engineering design that allows controlling safety factors and failure probabilities with respect to different modes of failure. Since failure probabilities are very sensitive to tail assumptions, and safety factors can be insufficient, a double check for the safety of the engineering structure is done. The dual method uses an iterative process that consists of repeating a sequence of three steps: (a) an optimal (in the sense of optimizing an objective function) classical design, based on given safety factors, is done, (b) failure probabilities or bounds of all failure modes are calculated, and (c) safety factors bounds are adjusted. The three steps are repeated until convergence, i.e. until the safety factors lower bounds and the mode failure probability upper bounds are satisfied. In addition, a sensitivity analysis of the cost and reliability indices to the data parameters is done. The proposed method is illustrated by its application to the design of a rubble-mound breakwater.

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## 1 Introduction

The phases that an engineering structure undergoes are: construction, useful life, maintenance and repair, dismantling, etc. Each phase has an associated duration. During each of these phases, the structure and the environment experiment a continuous sequence of outcomes, that have to be analyzed in the project

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(Ditlevsen (1997)). The objective of the project design is to verify that the structure fulfills the project requirements during these phases. Initially, the engineer must decide the duration of the useful life of the work being designed. This duration determines the resulting design.

Next, the modes of failure of the structure must be defined. A mode describes the form or mechanism in which the failure of a part of the structure or one of its elements is produced. Each mode of failure is defined by its corresponding verification (non-failure) equation, that admits different representations as, for example:

$$g_i^*(x_1, x_2, \dots, x_n) = \frac{h_{si}(x_1, x_2, \dots, x_n)}{h_{fi}(x_1, x_2, \dots, x_n)} - 1 > 0 \tag{1}$$

where $h_{si}(x_1, x_2, \dots, x_n)$ and $h_{fi}(x_1, x_2, \dots, x_n)$ are two opposing magnitudes (as stabilizing to overturning forces, strengths to ultimate stresses, etc.) that avoid and produce the associated mode of failure, respectively, $i$ refers to the mode of failure, and $(x_1, x_2, \dots, x_n)$ are the values of the variables involved. Checking whether or not this equation is satisfied, the safety of the structure with respect to such a mode of failure can be determined. If equation (1) holds, the failure does not occur; otherwise it occurs.

This check can be done from two different points of view, denoted here as (1) classic or deterministic, and (2) probability based.

In the former, and since an engineering design cannot be strictly safe, verification equations cannot be used for design. So, they are modified to increase safety and this leads to the safety constraint:

$$\frac{h_{si}(x_1^d, x_2^d, \dots, x_n^d)}{h_{fi}(x_1^d, x_2^d, \dots, x_n^d)} - F > 0; \quad F > 1$$

where $x_1^d, x_2^d, \dots, x_n^d$ are the design values of the variables $(X_1, X_2, \cdots, X_n)$, and $F$ is the safety factor associated with the mode of failure. Thus, in a classic design the design equations or constraints are written in terms of safety factors.

In the case of climatic actions, $x_1^d, x_2^d, \dots, x_n^d$ can be obtained from the state variables $H_s$ and $\overline{T}_z$. Definition of these state variables requires a stochastic model.

Safety factors have the advantage of being easily interpretable in terms of their physical or engineering meaning, but have the inconvenience of not giving a clear information on the reliability of the structure.

In the probabilistic based design, a joint probability $f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)$ is assumed for all the random variables involved, and the engineer calculates the probabilities of the different modes of failure:

$$P_{f_i} = \int\limits_{g_i^*(x_1,x_2,\dots,x_n) \le 0} f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n. \tag{2}$$

If the design variables lead to reasonable probabilities of these occurrences, i.e. probabilities below given upper bounds, the design is said to be safe. The main advantage of probabilistic based design is that the reliability of the structure can be evaluated if statistics are well defined. However, they have the shortcoming of being very sensitive to tail assumptions, and in some cases, as, for example, rubble-mound stability, runup, overtopping, geotechnical stability, etc., parametric dependencies and statistics are difficult to define.

In this paper the choice of the density $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ is not discussed. We assume it to be known.

Classic and probabilistic designs consider the useful life in a different manner, the former by selecting the "extreme" sea state descriptors ($H_s$ and $\overline{T}_z$) expected to occur during the expected life, and the latter by considering the random distribution of all random variables during the useful life.

Generally speaking an engineering design follows an iterative scheme, starting with an initialization phase, in which the design variables are chosen, followed by the verification of whether the safety factor or probability constraints are satisfied. If they are not, the engineer modifies his/her design variables and checks again. The process stops once either the safety factor or the safety probability constraints are fulfilled.

However, the process is usually stopped as soon as a reasonable solution is obtained. In the last years both methods have been improved by applying optimization techniques. The main advantage is that optimization techniques lead to optimal design and automation, i.e. the design variables are chosen by the optimization procedure and not by the engineer. His/her concerns are only the constraints to be imposed to the problem and the objective function to be optimized. There are works on design based on safety factors, failure probabilities and optimization techniques, and some combine them (see, for example, USACE, PIANC, IWR, etc.). An interesting combination is the one recommended by Vrouwenvelder (2002) in the frame of code calibration. In this paper, safety and reliability factors are combined with optimization procedures in a different way.

Furthermore, some verification equations are linked to well known and historically proved safety factors, whereas others have to fulfill some constraints specified by a probability of failure. In the present state of the art, it seems not only reasonable but convenient, using the advantages of both design paradigms avoiding the never ended dispute about the prevalence of one or the other.

The authors intention in this paper is not to provide an exhaustive treatment of all modes of failure of a rubble-mound breakwater, with a careful definition of the corresponding verification equations, nor giving methods for selecting the characteristic values of the design sea state. These two problems have been dealt with in many other papers. The aim of this paper is twofold: (a) present a dual (classic-probabilistic) design method based on safety factors and failure probabilities, i.e. able to combine the advantages of both design paradigms, together with optimization procedures (see ROM (2001), and Castillo et al. (2002)), and (b) describe how sensitivity analysis can be easily done.

The importance of both contributions must be clarified. The dual approach implies a double safety control. If the selected safety factors are not reasonable (too low or too high), the failure probabilities will point this out clearly, by their associated very high or very low values, respectively. Similarly, if failure probabilities are high or low, or the probabilistic assumptions are unrealistic, the associated safety factors will reveal these errors. For this reason, we recommend both controls, though the ideal situation will lead to equivalent constraints of safety factors and failure probabilities.

The sensitivity analysis will provide an excellent information of how much a small change in the parameters or assumptions (data) modifies the resulting design (geometric dimensions, costs, reliabilities, etc.). This will be useful to: (a) the designer, (b) the construction engineer, and (c) the code designer (he/she will know, for example, how much a reduction in the required safety factors or reliability indices increases the cost).

In the Coastal Engineering Manual of the USACE (U.S.A. Corps of Engineers) and in the final report of the MAST (Marine Science and Technology Programme) III Proverbs project (Probabilistic Design Tools for Vertical Breakwaters) of the European Union (1996-1999) the design of vertical breakwaters including the various aspects of hydraulic loading and foundation and structural strength is addressed. The main dealing of the report is the modelling of physical processes relevant for this type of structures as well as an overall probabilistic design concept in which the physics is integrated. Two alternative methods for including probabilistic analysis into the design process are given; the first of which is an optimization framework to find the optimal safety level and the second is a partial safety factor approach. In this paper a third alternative method, the traditional global safety coefficient together with the other two is considered, and the optimization framework is extended to the joint fulfillment of the most restrictive condition, probability of failure or global safety factors. Thus, the present paper should be considered as a continuation of the important research developed under the framework of USACE and the MAST III project.

The paper is structured as follows. In Section 2 the optimal classic and probabilistic design paradigms are described, where the term paradigm is used here to refer to patron, i.e. a way of facing and solving the engineering design problem. In Section 3 the proposed optimal dual design paradigm is presented. In Section 4 a technique for sensitivity analysis, based on the duals of the primal mathematical optimization problems associated with safety factors and failure probabilities, is explained. Section 5 illustrates the proposed method using one example of application dealing with the design of a rubble-mound breakwater.

Finally, Section 6 gives some conclusions.

## 2 The Optimal Classic and Probabilistic Paradigms

In this section the two paradigms that are used in engineering design: the classic or safety factor based, and the probabilistic paradigms, are described.

**Safe and failure domains.** In the design and reliability analysis of a maritime work, there are some random variables $(X_1, \ldots, X_n)$ involved. They include geometric variables, material properties, loads, etc. In this paper we use uppercase letters to refer to random variables, and the corresponding lowercase letters to refer to particular instantiations of these variables. They belong to an $n$-dimensional space, which, for each mode of failure, can be divided into two regions, the safe and the failure regions:

$$\left.\begin{array}{ll}\text{Safe Region:} & \mathcal{S} \equiv \{(x_1, x_2, \ldots, x_n)\}|g_i(x_1, x_2, \ldots, x_n) \geq 1\} \\ \text{Failure Region:} & \mathcal{F} \equiv \{(x_1, x_2, \ldots, x_n)\}|g_i(x_1, x_2, \ldots, x_n) < 1\}\end{array}\right\}; \quad i \in I \tag{3}$$

where $I$ is the set of all modes of failure, and $g_i(x_1, x_2, \ldots, x_n) = g_i^*(x_1, x_2, \ldots, x_n) + 1$.

Since the constraint $g_i(x_1, x_2, \ldots, x_n) = 1$ defines strict stability or security, to increase safety, the constant 1 is normally replaced by a larger constant called safety factor $F_i^0$. Then, we have

$$\text{Design Region in mode i:} \quad \mathcal{S}_i \equiv \{(x_1, x_2, \ldots, x_n)\}|g_i(x_1, x_2, \ldots, x_n) \geq F_i^0\}; \quad i \in I \tag{4}$$

It is important to distinguish between design values of the random variables $X_i : i = 1, 2, \cdots, n$, which in this paper are assumed to be the expectations or the characteristic values, and denoted $\tilde{x}_i$, and actual values $x_i$ (those existing in reality). Some of these design values are chosen by the engineer or given by the design codes, and some (associated with the design variables) are selected by the optimization procedure to be presented. In this paper, the set of basic variables $(X_1, \ldots, X_n)$ will be partitioned in five sets:

- $\mathbf{d}$: design or geometric variables,
- $\boldsymbol{\eta}$: the set of parameters used in the classic design,
- $\boldsymbol{\phi}$: The set of basic random variables used only in the probabilistic design,
- $\boldsymbol{\kappa}$: the set of parameters used in the probabilistic design, defining the random variability and dependence structure of the random variables involved,
- $\boldsymbol{\psi}$: the auxiliary (non-basic) variables which values can be obtained from those of the basic variables, using some formulae.

The corresponding mean or characteristic vectors will be denoted $\tilde{\mathbf{d}}$, $\tilde{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{\phi}}$, respectively. Initially all the variables can be assumed random, however, some of them are assumed to be deterministic (their random character does not have an important influence on the reliabilities), i.e. especial cases of random variables.

As an illustration, for the rubble-mound breakwater example, we have (see the Appendix B for the notation):

$$\mathbf{d} = (a, b, c, d, e, F_c, f, n, p, q, r, s, t, \alpha_\ell, \alpha_s) \tag{5}$$

$$\boldsymbol{\eta} = (C_{al}, C_c, C_{co}, C_{ul}, D_{W_L}, g, \gamma_c, \gamma_s, \gamma_w) \tag{6}$$

$$\boldsymbol{\phi} = (A_r, A_u, B_r, B_u, C_f, H, T, \mu_c) \tag{7}$$

$$\boldsymbol{\kappa} = (H_s, \overline{T}_z, v_{A_r}, v_{B_r}, v_{C_f}, \mu_{A_r}, \mu_{B_r}, \mu_{A_u}, \mu_{B_u}, \mu_{C_f}, \mu_{\mu_c}, \sigma_{A_u}, \sigma_{B_u}, \sigma_{\mu_c}) \tag{8}$$

$$\boldsymbol{\psi} = (A_c, F_h, F_v, I_r, I_{r_0}, L, \ell, \ell_e, P_{S_0}, R, R_u, S_0, V_c, V_1, V_2, V_3, W, W_1, W_f, \alpha, \lambda, \phi_e) \tag{9}$$

Note that only the basic variables $\boldsymbol{\phi}$ (and the corresponding auxiliar non-basic variables) have been assumed to be random in this example.

## 2.1 The optimal safety factor paradigm

Since one is not interested in strict failure, but in having some sufficiently safe maritime works, design is based on the condition

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \geq F_i^0;\ \forall i \in I. \tag{10}$$

A set of values $(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}})$ is said to be a classic design, if and only if it satisfies (10). This set is also an optimal classic design if and only if it

$$\underset{\tilde{\mathbf{d}}}{\text{Minimizes}} \quad Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}) \tag{11}$$

where $Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}})$ is the function to be optimized (cost function, for example), subject to

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \geq F_i^0;\ \forall i \in I. \tag{12}$$

## 2.2 The optimal probabilistic paradigm

Given a set of values of the design variables $\tilde{\boldsymbol{\eta}}$, the probability of failure, $P_{f_i}$, can be calculated using the joint probability density function

$$f(\mathbf{x}) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \boldsymbol{\theta}) \tag{13}$$

of all variables involved, where $\boldsymbol{\theta}$ is a parametric vector, by means of the integral:

$$P_{f_i} = \int\limits_{g_i(x_1,x_2,\dots,x_n)\le 1} f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n;\boldsymbol{\theta})dx_1dx_2\dots dx_n. \tag{14}$$

In this paper we assume that the parametric vector $\boldsymbol{\theta} = (\tilde{\mathbf{d}},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\phi}},\boldsymbol{\kappa})$, i.e. it contains the mean or the characteristic variables $\tilde{\mathbf{d}},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\phi}}$ and some other extra vector of parameters $\boldsymbol{\kappa}$ (for defining the variable dispersions, the dependence structure, etc.).

A set of values $\tilde{\mathbf{d}}$ for the design variables is said to be a probability based design if and only if it satisfies

$$P_{f_i} \le p_i^0;\ \forall i \in I. \tag{15}$$

where $p_i^0; i \in I$ are the upper bounds for the corresponding probabilities of failure.

This set is also an optimal probabilistic design if and only if it

$$\underset{\tilde{\mathbf{d}}}{\text{Minimizes}} \quad Q\ (\tilde{\mathbf{d}},\tilde{\boldsymbol{\eta}}) \tag{16}$$

subject to

$$P_{f_i}(\tilde{\mathbf{d}},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\phi}},\boldsymbol{\kappa}) \le p_i^0;\ \forall i \in I. \tag{17}$$

Since the probability of failure must be a small number, the engineer is presented with a tail probability problem. It is well known that tail probabilities are very sensitive to the assumed model (see Galambos (1987), Castillo (1988a) and Castillo, Solares and Gómez (1996,1997a,1997b)). Thus, the model selection must be done with care.

Unfortunately, calculation of $P_{f_i}(\tilde{\mathbf{d}},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\phi}},\boldsymbol{\kappa})$ is very difficult. So, to eliminate the need of complex numerical integrations, the "First Order Reliability Methods" (FORM) transform the initial set of variables into an independent multinormal set and use a linear approximation. These methods appeared in the field of structural reliability with Freudenthal (1956), and have been expanded by Hasofer and Lind (1974), Rackwitz and Flessler (1978), Breitung (1984), Wirsching and Wu (1987) and Wu, Burnside and Cruse (1989), etc. They have shown to give precise results and have demonstrated to be much more efficient than Monte Carlo simulation techniques for estimating extreme percentiles (see, for example, Wirsching (1987), or Haskin, Staple and Ding (1996)). For a complete description of some of these methods and some illustrative examples see Ditlevsen and Madsen (1996) and Madsen, Krenk and Lind (1986).

It is well known that the calculation of the probabilities of failure against a mode is equivalent to solving the following non linear programming problem

$$\underset{\mathbf{d},\boldsymbol{\eta},\boldsymbol{\phi}}{\text{Minimize}} \quad \beta_i = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{18}$$

subject to

$$g_i(\mathbf{z}, \mathbf{d}, \boldsymbol{\eta}, \boldsymbol{\phi}, \boldsymbol{\kappa}) = 1 \tag{19}$$

$$T(\mathbf{d}, \boldsymbol{\eta}, \boldsymbol{\phi}; \boldsymbol{\theta}) = \mathbf{z} \tag{20}$$

Since the probabilities of failure are very difficult to calculate, the problem (16)-(17) can be replaced by the problem

$$\underset{\tilde{\mathbf{d}}}{\text{Minimize}} \quad Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}) \tag{21}$$

subject to

$$\beta_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}, \boldsymbol{\kappa}) \geq \beta_i^0; \ \forall i \in I, \tag{22}$$

where $\beta_i^0; i \in I$ are the allowed lower bounds for the corresponding $\beta$-values.

The problem (21)-(22) cannot be solved by standard procedures because the evaluation of the $\beta$-values involves the optimization problem (18)-(19). However, other methods, using two nested optimization algorithms can be found in the literature.

### 2.3 Equivalence of classic and probability based procedures

It is interesting to point out that for practically all real cases, both methods, the classic, (11)-(12), and the probability based, (21)-(22), are equivalent, in the sense that they lead to the same design values, $\tilde{\mathbf{d}}$, for given $\tilde{\boldsymbol{\eta}}, \boldsymbol{\kappa}$. In other words, given any set of safety factor bounds, there exist a set of $\beta$-values bounds such that the solution $\tilde{\mathbf{d}}$ of the problems (11)-(12) and (21)-(22) coincide, and vice versa.

It is important to understand this correspondence between safety factors and $\beta$-values for understanding the proposed dual method in depth.

## 3 The Optimal Dual Design Paradigm

To incorporate the advantages of both, the optimal classic and the probability designs, a dual method is presented and solved by an iterative scheme.

The dual problem is formulated as follows,

$$\underset{\tilde{\mathbf{d}}}{\text{Minimize}} \quad Q \ (\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}), \tag{23}$$

subject to

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \geq F_i^0; \ \forall i \in I, \tag{24}$$

and

$$\beta_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}, \boldsymbol{\kappa}) \geq \beta_i^0; \ \forall i \in I. \tag{25}$$

Unfortunately, this problem cannot be solved directly because the constraints (25) involve complicated integrals, optimization methods, or both. Thus, an alternative method is needed. The method presented in this section proceeds by an iterative method that consists of repeating a sequence of three steps: (1) an optimal (in the sense of optimizing an objective function) classic design, based on given safety factors, is done, (2) failure probabilities (reliability indices) or bounds against all failures modes are determined, and (3) all mode safety factor bounds are adjusted. The three steps are repeated until convergence, i.e. until the safety factors lower bounds and the mode of failure probability upper bounds are satisfied. More precisely, the method proceeds as follows:

**Step 1. Solving the optimal classic design.** An optimal classic design based on the actual safety factors, that are fixed initially to their corresponding lower bounds, is done. In other words, the following problem is solved

$$\underset{\tilde{\mathbf{d}}}{\text{Minimize}} \quad Q\ (\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}), \tag{26}$$

subject to

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \geq F_i^k;\ \forall i \in I. \tag{27}$$

where $k$ refers to the $k$ iteration.

The result of this process is a set of values of the design variables, that satisfy the safety factor constraints (27).

**Step 2. Evaluating new $\beta$-values.** The actual $\beta$-values associated with all modes of failure are evaluated, or upper bounds for them determined, based on the values of the design projects obtained in Step 1. To this end, the problem

$$\underset{\mathbf{d}, \boldsymbol{\eta}, \boldsymbol{\phi}}{\text{Minimize}} \quad \beta_i^k = \sqrt{\sum_{j=1}^{n} z_j^2} \tag{28}$$

subject to

$$g_i(\mathbf{z}, \tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}, \boldsymbol{\kappa}) = 1 \tag{29}$$

$$T(\mathbf{d}, \boldsymbol{\eta}, \boldsymbol{\phi}; \boldsymbol{\theta}) = \mathbf{z} \tag{30}$$

where $T(\mathbf{d}, \boldsymbol{\eta}, \boldsymbol{\phi}; \boldsymbol{\theta})$ is the well known transformation to standard independent normal variables, is solved for any $i \in I$, i.e. in this step as many optimization problems as the number of modes of failure are solved.

**Step 3: Updating safety factor values.** The safety factors bounds are adequately updated for the actual safety factors and the actual $\beta$-values to satisfy the required bounds. To this end, the safety factors bounds are modified using the increments

$$\Delta F_i = \rho(\beta_i^0 - \beta_i^k); \quad i \in I, \tag{31}$$

where $\rho$ is a small positive constant. Note that there is a monotone relation between safety factors and reliability coefficients. This justifies the use of Equation (31) to update safety factor bounds.

To avoid large increments of the safety factors in each iteration, the value of $\rho$ can be selected using the expression

$$\rho = \min\left(\rho_0, \min_i \left(\frac{\Delta}{|\beta_i^0 - \beta_i^k|}\right)\right) \tag{32}$$

where $\Delta$ is a small quantity, for example, $\Delta = 1$, and $\rho_0$ is a small number. In addition, if, using this formula, some safety factor becomes smaller than the associated lower bound $F_i^k$, it is kept equal to $F_i^0$. Note that the actual safety factors need to be calculated using

$$F_i = g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \tag{33}$$

because the values $F_i^0$ are only lower bounds, but not actual values.

**Comments** The result of the iterative scheme is an optimal design that satisfies the required safety factors and $\beta$-values bounds. The optimal classic design and the optimal probability based design are particular cases of the present dual design procedure. If a classic design is looked for, large bounds for the probabilities of failure $(\beta_i^0; \forall i \in I)$ must be selected. If, on the contrary, a probability based design is looked for, low safety factors lower bounds $(F_i^0 = 1; \forall i \in I)$ must be chosen.

In fact, the proposed method can be extended to include global and partial safety factors (see Rackwitz (1997)). The authors are working in this line that is the aim of another paper.

To end these comments, we indicate that there exist alternative iterative methods to the one explained above (see Castillo, Conejo, Mínguez and Castillo (2003)).

# 4 Sensitivity Analysis

In this section it is explained how a sensitivity analysis can be done with practically no extra cost, using standard optimization packages. The method consists of: (a) writing the two optimization problems (26)-(27) and (28)-(29) in an equivalent form, (b) considering the associated dual problems, and (c) realizing that the values of some dual variables are the sensitivities looked for. The term dual in this section has a different meaning than the same term in previous sections. Here it refers to the duality relation used in mathematical optimization. Every primal minimization problem has a dual associated maximization problem, with related solutions. Duality here refers to this correspondence. Only the method for the optimization problem (26)-(27) is explained below, but the same technique applies to the problems (28)-(29).

The method is based on observing that the problem

$$\underset{\tilde{\mathbf{d}}}{\text{Minimizing}} \quad Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}) \ , \tag{34}$$

subject to

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}) \geq F_i; \ \forall i \in I. \tag{35}$$

is equivalent to the problem

$$\underset{\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}_0, \tilde{\boldsymbol{\phi}}_0}{\text{Minimizing}} \quad Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}_0) \ , \tag{36}$$

subject to

$$g_i(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}_0, \tilde{\boldsymbol{\phi}}_0) \geq F_i; \ \forall i \in I. \tag{37}$$

and

$$(\tilde{\boldsymbol{\eta}}_0, \tilde{\boldsymbol{\phi}}_0) = (\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\phi}}). \tag{38}$$

Since constraints (38) involve the data in their right hand sides, and the dual variables are the sensitivities of the objective function value to changes in the constraints right hand side terms, the desired sensitivities can be obtained by printing the values of the corresponding dual variables. In other words, the values of the dual variables associated with the constraints in (38), give how much the objective function $Q(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}})$ changes with a very small unit increment of the corresponding data parameter.

One of the interesting findings of the proposed procedure is that it informs the engineer of how the solution varies as the values of the project variables are changed.

# 5 Optimized Design of a rubble-mound Breakwater

In this section the proposed procedure is applied to the design of a rubble-mound breakwater. The main section of the breakwater is shown in Figure 1. The list of geometric variables and parameters appears in Appendix B. Notice that these parameters define geometrically the different elements of the cross section and must be defined in the construction drawings. Our goal is an optimal design based on minimizing the construction cost, subject to safety factor and failure probability constraints for each mode of failure. In fact the cost to be minimized should include maintenance, repairing, dismantling, etc.; however, for the sake of simplicity only the construction cost is considered here.

## 5.1 Modes of failure

Though other important modes of failure as, for example, armor structural integrity, toe instability and geotechnical failure or failure in subsoil, could be considered, for the sake of simplicity, only three modes of failure of the rubble-mound breakwater are included in the optimization process: (1) overtopping failure, (2) armor failure, and (3) crownwall sliding failure.

Some of those modes are correlated, because they have common agents, or because one mode can induce the occurrence of others. Only the correlation due to common agents is considered in this paper. Interaction between modes of failure is complex and poorly understood and surely it will be an active research area during the next years. This interaction is not considered in the breakwater example. Thus, the calculated reliability has to be observed as an incomplete estimate, not because of the tool but of the absence of appropriate information. The availability of the tool calls for good experimental work to provide such information.

Strictly speaking the extraction of pieces from the main layer of a mound breakwaters do not correspond to a classical structural failure. It should be noted that a mound is a granular system where geometrical connectivity and structural stress transmission occur by friction and interlocking between units. As usual, any mode of failure is related to a pre-established criterion, or threshold level, which gives an indication of the residual resistance of the structure before collapse. Like other disciplines of the civil engineering the occurrence of the failure does not necessarily mean that the structure collapse but that its resistance is seriously diminished and its functionality seriously affected.

For rubble-mound breakwaters, there are at least three criteria of failure: initiation of damage, Iribarren´s damage and destruction, (see Losada, 1990 for a more comprehensive discussion). 2-5% displacement of units is the typical criterion for initiation of damage. The adoption of one or other criterion for design, depends on several factors, among them, the capacity or not to undertake the necessary reparation, the economic and environmental consequences of the failure, etc.

**Overtopping failure.** For a rubble-mound breakwater of slope $\tan \alpha_s$ and freeboard $F_c$, (see Figure 1), and a given wave of height $H$ and period $T$, the volume of water that overtops the structure can be estimated from the volume of water that would rise over the imaginary extension of the slope exceeding the freeboard level. With this approximation, overtopping (failure) occurs whenever the difference between the maximum excursion of water over the imaginary slope, $R_u$, called wave run-up, exceeds the freeboard $F_c$.

Losada (1990) proposed the following verification equation based on experiments to evaluate the dimensionless quantity $R_u/H$:

$$\frac{R_u}{H} = A_u\left(1 - e^{B_u I_r}\right) \tag{39}$$

where $A_u$ and $B_u$ are coefficients that depend on the armor units, $\alpha_s$ is the seaside slope angle, and $I_r$ is the Iribarren number

$$I_r = \frac{\tan\alpha_s}{\sqrt{H/L}} \tag{40}$$

where $L$ is the wave length.

Under such conditions, the occurrence of failure can be verified from the following safety factor constraint:

$$\frac{F_c}{R_u} \geq F_o \tag{41}$$

where $F_o$ is the overtopping safety factor.

**Armor failure.** This failure refers to the removal of concrete cubes from the armor layer. Based on experiments, Losada (1990) proposed the following verification equation to evaluate the dimensionless quantity:

$$\frac{W}{\gamma_w H^3} = R\Phi_e \tag{42}$$

where $\gamma_w$ is the water unit weight, $\Phi_e$ is the stability function, $R$ is an adimensional constant, that depends on $\gamma_c$ and $\gamma_w$, and $W$ is the individual armor block weight, that are given by

$$\begin{aligned} W &= \gamma_c \ell^3 & (43)\\ R &= \frac{\gamma_c/\gamma_w}{\left(\dfrac{\gamma_c}{\gamma_w} - 1\right)^3} & (44)\\ \Phi_e &= A_r(I_r - I_{r_0})\exp[B_r(I_r - I_{r_0})] & (45)\\ I_r &\geq I_{r_0} & (46) \end{aligned}$$

where $\gamma_c$ is the concrete unit weight, $\ell$ is the characteristic block side dimension (two layers of pieces and random placement are assumed),

$$I_{r_0} = 2.656\tan\alpha_s, \tag{47}$$

and $A_r$ and $B_r$ depend on $\cot\alpha_s$ by the approximate experimental relations

$$\begin{aligned} A_r &= 0.2566 - 0.177047\cot\alpha_s + 0.0342933\cot^2\alpha_s & (48)\\ B_r &= -0.0201 - 0.4123\cot\alpha_s + 0.055\cot^2\alpha_s & (49) \end{aligned}$$

that are assumed to be valid for $1.5 \leq \cot\alpha_s \leq 3$.

Under such conditions, the occurrence of failure can be determined from the following safety factor constraint:

$$\frac{W}{\gamma_w R \Phi_e H^3} \geq F_a \tag{50}$$

where $F_a$ is the armor failure safety factor.

**Crownwall sliding failure.** This failure occurs when the crownwall slides with respect to its base due to water pressure forces. The sliding failure can be verified by the following verification equation (see Figure 2(b))

$$\mu_c(W_1 - F_v) = F_h \tag{51}$$

where $\mu_c$ is the friction coefficient, and (see Martin et al. (1999)):

$$\begin{aligned} F_h &= (S_0 - A_c)P_{S_0} + (W_f + A_c)\lambda P_{S_0} && (52)\\ F_v &= \frac{1}{2}\lambda P_{S_0} e && (53)\\ W_1 &= V_c\gamma_c - W_f e\gamma_w && (54)\\ V_c &= pq + se && (55) \end{aligned}$$

where $F_h$ and $F_v$ are the total vertical and horizontal forces due to water pressure, $S_0$ the wave height due to run-up, $A_c$ the berm level, $P_{S_0}$ the wave water pressure, $W_f$ the submerged height of the crownwall, $e$ the crownwall width, $V_c\gamma_c$ is the total crown concrete weight, and $W_1$ is the actual crown weight (dry and submerged parts).

For $I_r \geq 2$ the pressure forces acting on the crownwall at level $z$ are (see Figure 2):

$$P_d(z) = \begin{cases} \lambda P_{S_0} & if \quad z < A_c \\ P_{S_0} & if \quad z > A_c \end{cases} \tag{56}$$

where

$$\begin{aligned} I_r &\geq 2 && (57)\\ P_{S_0} &= \alpha\gamma_w S_0 && (58)\\ S_0 &= H\left(1 - \frac{A_c}{R_u}\right) && (59)\\ \alpha &= 2C_f\left(\frac{R_u}{H}\cos\alpha_s\right)^2 && (60)\\ \lambda &= 0.8\exp\left(-10.9\frac{d}{L}\right) && (61)\\ \left(\frac{2\pi}{T}\right)^2 &= g\frac{2\pi}{L}\tanh\left(\frac{2\pi D_{W_L}}{L}\right) && (62) \end{aligned}$$

where $g$ is the acceleration of gravity, $d$ the seaside berm width, $\alpha$ a random non-dimensional variable, $D_{W_L}$ the design water level, and $C_f$ is an experimental random coefficient.

Notice that only the first peak of pressure is verified; reflected pressure can be worked out in the same fashion (see Martin et al. (1999) for details). The sliding failure can be verified by the following safety factor constraint

$$\frac{\mu_c(W_1 - F_v)}{F_h} \geq F_s \tag{63}$$

where $F_s$ is the sliding safety factor.

## 5.2 Practical design criteria

In maritime works there are some rules of good practice that should be observed. Some of them are country dependent and some have historical roots. Those used in this example, are the following (see Figure 1). Bullets after an inequality indicate that they become active, i.e. they degenerate to strict equalities, in the numerical example to be explained.

1. **Layers and berm widths**:

$$a = 2\ell; \;\; b = 2\ell_e; \;\; d \geq 2\ell \;(\bullet); \;\; f \geq 2\ell_e \;(\bullet). \tag{64}$$

2. **Filter conditions**:

$$\frac{W}{20} \leq \ell_e^3\gamma_s \;(\bullet); \;\; \ell_e^3\gamma_s \leq \frac{W}{10} \tag{65}$$

3. **Construction reasons**:

$$b + c \leq D_{W_L}; \;\; D_{W_L} \leq a + b + c; \;\; t = 1m; \;\; r = 2t; \;\; p = 2 \tag{66}$$

4. **Economic reasons**:

$$1.5 \leq \cot\alpha_s \leq 3; \;\; (\bullet)1.5 \leq \cot\alpha_\ell \leq 3 \tag{67}$$

5. **Wave breaking and overtopping control**:

$$A_c \geq 3H_s/4 \;; \;\; F_c \geq S_0 \tag{68}$$

6. **Geometric identities**:

$$D_{W_L} + F_c = b + c + s + q; \;\; W_f = s + q - F_c; \;\; A_c = a + b + c - D_{W_L} \tag{69}$$

7. **Model requirements**:

$$s \leq 12(\bullet); \;\; 2 \leq q \leq 8(\bullet); \;\; e \geq 10 \tag{70}$$

where $\ell$ and $\ell_e$ are the equivalent cubic block side for the main layer and the secondary layers, respectively, and $H_s$ is the significant weight height, statistical descriptor of the sea state.

## 5.3 Cost function

The details of the derivation of the cost function are given in the Appendix B. The resulting total construction cost becomes:

$$C = C_c V_c + C_{al} V_1 + C_{ul} V_2 + C_{co} V_3 \tag{71}$$

where $V_c, V_1, V_2$ and $V_3$ are the concrete, armor layer, underlayer, and core volumes, respectively, and $C_c, C_{al}, C_{ul}$ and $C_{co}$ are the respective construction costs per unit volume.

## 5.4 Random and deterministic project factors

In this example the useful life of the rubble-mound breakwater is $D = 50$ years. To apply the proposed model, the set of deterministic and random variables need to be defined.

**Parameters**:

1. Geometric parameters:

   (a) The freeboard, $F_c$, and the slopes $\alpha_s$ and $\alpha_\ell$ are given by their nominal deterministic values.

2. Material properties parameters:

   (a) Unit weights $\gamma_c, \gamma_w$ and $\gamma_s$ of the concrete, water and quarry stone are given by their nominal deterministic values.

3. Mechanical properties parameters:

   (a) The friction factor $\mu_c$ is a normal random variable with mean $\mu_{\mu_c} = 0.6$ and characteristic 0.05 value $\mu_{c_{0.05}} = 0.55$.

4. Experimental parameters:

   (a) Runup model

$$\frac{R_u}{H} = A_u \left(1 - e^{B_u I_r}\right) \tag{72}$$

   - $A_u$ is a normal random variable $N(\mu_{A_u}, \sigma^2_{A_u})$, where $\mu_{A_u} = 1.05$ and $\sigma_{A_u} = 0.21$.
   - $B_u$ is a normal random variable $N(\mu_{B_u}, \sigma^2_{B_u})$, where $\mu_{B_u} = -0.67$ and $\sigma_{B_u} = 0.134$.

     These two variables are assumed to be independent, even though they have been determined by a fitting procedure, because the resulting correlation was negligible.

(b) Armor stability model

$$\Phi_e = A_r(I_r - I_{r_0})\exp[B_r(I_r - I_{r_0})] \tag{73}$$

- $A_r$ is a normal random variable $N(\mu_{A_r}, \sigma^2_{A_r})$, where

$$\mu_{A_r} = 0.2566 - 0.177047\cot\alpha_s + 0.0342933\cot^2\alpha_s; \quad v_{A_r} = 0.15 \tag{74}$$

- $B_r$ is a normal random variable $N(\mu_{B_r}, \sigma^2_{B_r})$, where

$$\mu_{B_r} = -0.0201 - 0.4123\cot\alpha_s + 0.055\cot^2\alpha_s; \quad v_{B_r} = 0.15 \tag{75}$$

5. Crown stability model

(a) $C_f$ is a normal random variable with mean $\mu_{C_f}$ and standard deviation $\sigma_{C_f}$.

(b) $\alpha$ is a random variable which distribution can be derived from Equation (60).

(c) $\lambda$ is given by its nominal deterministic value.

(d) $S_0$ is a random variable which distribution can be derived from Equation (59).

(e) $P_d$ and $P_{S_0}$ are random variables which distributions can be derived from Equation (56) and (58), respectively.

**Agents**:

1. Climatic parameters:

(a) The astronomical tide $h_1$ is assumed to be a uniform random variable $U(zp, zp + tr)$, where $zp$ is the zero port and $tr$ is the tidal range.

(b) The meteorological tide $h_2$ is assumed to be a normal random variable $N(\mu_{h_2}, \sigma^2_{h_2})$. Then, the water level is $D_{W_L} = h_1 + h_2$.

(c) The maximum wave height $H_D$ and period $T_p$ in a sea state are random variables with cumulative distribution functions:

$$\begin{aligned} F_{H_D}(x; H_s) &= 1 - \exp\left[-\left(\frac{x - 1.263 - 0.326H_s - 0.172H_s^2}{1.465}\right)^{2.507}\right]; \\ &\qquad x \geq 1.263 + 0.326H_s + 0.172H_s^2 \end{aligned} \tag{76}$$

and

$$\begin{aligned} F_{T_p}(x; H_s) &= \exp\left[-\left(\frac{11.176 + 3.756H_s - 0.415H_s^2 - x}{7.597}\right)^{2.992}\right]; \\ &\qquad x \leq 11.176 + 3.756H_s - 0.415H_s^2 \end{aligned} \tag{77}$$

where $H_s$ is a random variable with exponential probability density function:

$$f_{H_s}(x) = 1.58247 \exp(-1.58247(x-3)); \quad x \geq 3;$$

where only sea states with $H_s > 3$ have been considered.

These distributions of $H_D$ and $T_p$ as a function of $H_s$, and the distribution of $H_s$ have been derived, based on Gijón buoy data. The data corresponds to 5.69 years, 260 storms (significant wave height > 3m), and 1857 (one hour) sea states.

**Dependence assumptions**:

All the above random variables are assumed to be independent, and all other variables are assumed to be deterministic. It is important to explain here that the correlation of the different modes of failure comes from the fact that they depend on common variables. Thus, even in the case of assuming independent variables, the modes of failure will become correlated. In other words, the main source of mode of failure correlation is its dependence on common variables and not the dependence of the variables themselves.

The above probability functions and the value of their parameters have been chosen just for illustration purposes. For applying the method to real cases, a more careful selection has to be done, using long term data records. Only a few countries have enough information to infer these functions adequately.

## 5.5 Formulation and solution of the dual problem

Following the process described in Section 4, firstly the safety factor lower bounds $F_o^0, F_s^0$ and $F_a^0$, and the reliability indices lower bounds $\beta_o^0, \beta_s^0$ and $\beta_a^0$, are chosen; next the iteration process is repeated until all the constraints are satisfied. The three steps are as follows.

**Step 1: Optimal classic design.** For iteration $k$, in the classic design the construction cost function is minimized

$$C(\tilde{\mathbf{d}}, \tilde{\boldsymbol{\eta}}) \tag{78}$$

subject to the safety factor constraints:

*Overtopping failure:*

$$\frac{F_c}{R_u} \geq F_o^k \tag{79}$$

*Armor failure:*

$$\frac{W}{\gamma_w R \Phi_e H^3} \geq F_a^k \tag{80}$$

*Crownwall sliding failure:*

$$\frac{\mu_c(W_1 - F_v)}{F_h} \geq F_s^k \tag{81}$$

and the design constraints.

**Step 2: Evaluation of the reliability indices.** Since in this paper we deal with the probability of failure during the useful live of the structure, the probability of failure for mode $i$ is:

$$P_{f_i} = 1 - \left(1 - \int_3^\infty f_{H_s}(H_s) P_{f_{ss_i}}(H_s) dH_s\right)^{n_{ss}D} \tag{82}$$

where $P_{f_{ss_i}}(H_s)$ is the probability of failure for mode $i$ in a sea state defined by $H_s$, and $n_{ss}$ is the equivalent number of sea states per year ($n_{ss} = 326$ for the Gijon's data) and $D$ is the useful life.

Note that the equivalent number of sea states allows taking into consideration the dependence of events in sea states. A number $n_{ss}$ smaller than the actual number of sea states indicates dependence, and a number coincident with it, implies the independence assumption.

This simplifying assumption has a strong theoretical basis, because it is well known (see Coles (2001) (page 96)) that $m$-dependence (dependence for close events but independence for separate events) in a stationary series leads to a change in the equivalent period length (which is reduced as dependence increases). This implies that $m$-dependence can be ignored when the parameters are estimated from real (dependent) samples.

Integral (82) can be evaluated using Gauss-Legendre's cuadrature formula as:

$$P_{f_i} \approx 1 - \left(1 - \frac{H_s^{max} - H_s^{min}}{2} \sum_{j=1}^{n} w_j f_{H_s}(H_{s_j}) P_{f_{ss_i}}(H_{s_j})\right)^{n_{ss}D} \tag{83}$$

where $w_j$ and $H_{s_j}$ are the Gauss weights and and points, respectively.

To calculate $P_{f_{ss_i}}(H_{s_j})$ the following procedure is used for each $H_{s_j}$:

1. Variables are transformed into independent unit normals:

$$\begin{array}{rcl}
u_1 &=& \Phi\left((A_u - \mu_{A_u})/\sigma_{A_u}\right) = \Phi(z_1) \\
u_2 &=& \Phi\left((B_u - \mu_{B_u})/\sigma_{B_u}\right) = \Phi(z_2) \\
u_3 &=& \Phi\left((A_r - \mu_{A_r})/\sigma_{A_r}\right) = \Phi(z_3) \\
u_4 &=& \Phi\left((B_r - \mu_{B_r})/\sigma_{B_r}\right) = \Phi(z_4) \\
u_5 &=& F_{T_p}(T; H_{s_j}) = \Phi(z_5) \\
u_6 &=& F_{H_D}(H; H_{s_j}) = \Phi(z_6) \\
u_7 &=& \Phi((\mu_c - \mu_{\mu_c})/\sigma_{\mu_c}) = \Phi(z_7) \\
u_8 &=& \Phi((C_f - \mu_{C_f})/\sigma_{C_f}) = \Phi(z_8) \\
u_9 &=& (h_1 - zp)/tr = \Phi(z_9) \\
u_{10} &=& \Phi((h_2 - \mu_{h_2})/\sigma_{h_2}) = \Phi(z_{10})
\end{array} \tag{84}$$

2. Three different optimization problems are solved for each $H_{s_j}$:

$$\underset{A_u, B_u, H, T, A_r, B_r, C_f, \mu_c, h_1, h_2}{\text{Minimize}} \quad {\beta_{i,j}}^2 = \sum_{k=1}^{10} z_k^2 \tag{85}$$

subject to (84) and one of the following failure mode constraints:

**Overtopping failure:**

$$1 = \frac{F_c}{R_u} \tag{86}$$

**Armor failure:**

$$1 = \frac{W}{\gamma_w R \Phi_e H^3} \tag{87}$$

**Crownwall sliding failure:**

$$1 = \frac{\mu_c (W_1 - F_v)}{F_h} \tag{88}$$

3. The probability of failure for mode $i$ in a sea state defined by $H_{s_j}$ is calculated using:

$$P_{f_{ss_i}}(H_{s_j}) = \Phi(-\beta_{i,j}) \tag{89}$$

Once the probabilities of failure for the different significant waves are calculated, the yearly probability of failure for mode $i$ is evaluated through the formula (82), and their associated $\beta$-values using

$$\beta_i^k = -\Phi^{-1}(P_{f_i}) \tag{90}$$

**Step 3: Testing convergence and updating safety factors.** If the obtained $\beta^k$-values satisfy the desired $\beta^0$-bounds the process is stopped and the optimal design is the one resulting from this iteration; otherwise, the safety factors are updated applying Equations (31) and (32) and the process continues with Step 1.

## 5.6 Numerical solution and analysis of the results

The proposed method has been implemented in GAMS (General Algebraic Modelling System) (see Castillo, Conejo, Pedregal, García and Alguacil (2001)), and the automatic optimal design of a rubble-mound breakwater with the following characteristics has been performed:

1. The target safety factors and reliability indices must depend on the consequences associated with the corresponding failure modes and the useful life of the structure being designed (a period of fifty years has been considered) (see Vrouwenvelder (2002)). Then, the following initial ($k = 0$) safety factor and $\beta$ reliability indices lower bounds have been chosen:

$$F_o^0 = 1.05; \quad F_s^0 = 1.5; \quad F_a^0 = 1.5, \tag{91}$$

$$\beta_o^0 = 2.32; \ \ \beta_s^0 = 3.08; \ \ \beta_a^0 = 3.08. \tag{92}$$

Note that the $\beta$ values correspond to failure probabilities of $0.01, 0.001$ and $0.001$, respectively.

2. For the safety factor design a significant wave height of $H_s = 6.5$ and the wave height and period corresponding to the 0.99 quantiles obtained from (76) and (77) have been selected. The other random variables are fixed to their mean values whereas $D_{W_L} = zp + tr$.

3. The nominal values, statistical and cost parameters used in this numerical example are:

   *Design values:*

$$p = 2m; \ \ r = 2m; \ \ t = 1m; \ \ D_{W_L} = 25m;$$

$$\gamma_c = 23.5KN/m^3; \ \ \gamma_s = 26KN/m^3; \ \ \gamma_w = 10.25KN/m^3$$

   *Statistical Properties:*

$$\mu_{A_u} = 1.05; \ \ \sigma_{A_u} = 0.21; \ \ \mu_{B_u} = -0.67; \ \ \sigma_{B_u} = 0.134; \ \ \mu_{\mu_c} = 0.6; \ \ \sigma_{\mu_c} = 0.01941; \ \ \mu_{C_f} = 1.45$$

$$v_{C_f} = 0.1; \ \ \mu_{A_r} = 0.2566 - 0.177047 \cot \alpha_s + 0.0342933 \cot^2 \alpha_s; \ \ v_{A_r} = 0.15;$$

$$\mu_{B_r} = -0.0201 - 0.4123 \cot \alpha_s + 0.055 \cot^2 \alpha_s; \ \ v_{B_r} = 0.15; \ \ \mu_{h_2} = 0.02414; \ \ \sigma_{h_2} = 0.11597$$

   *Cost Parameters:*

$$C_{al} = 818.4 euros/m^3; \ \ C_{ul} = 18.72 euros/m^3; \ \ C_{co} = 2.4 euros/m^3; \ \ C_c = 60.1 euros/m^3$$

**Analysis of results.** Table 1 shows the convergence of the process that is attained after 14 iterations. The first column shows the values of the geometric parameters, the safety factors and the corresponding reliability indices for the safety factor design, i.e. those corresponding to the safety factors in (91) without any consideration of the failure probabilities (reliability indices in (92)). However, since the resulting reliability indices are smaller than those in (92), the safety factors have been increased using (31) and (32), and the safety factor design repeated, in iteration 2. Since the same occurs in the subsequent iterations, the process is continued until iteration 14, where the constraints (92) are finally satisfied.

The last column of the table shows the values of the design variables, together with the safety factors and associated $\beta$-values. The active values appear underlined in this table. Note that the active role of the safety factors is induced by the reliability indices.

From this table one can conclude the following. The safety factor bounds $F_o^0, F_s^0$ and $F_a^0$ and the $\beta_o^0$ are inactive, while the $\beta_s^0$ and $\beta_a^0$ bounds are active. This implies that the $\beta_s^0$ and $\beta_a^0$ bounds are more stringent than the corresponding safety factor bounds, and that the safety against overtopping is implied by the sliding and armor failure safety constraints.

The sensitivities for the illustrative example are given in Tables 2 and 3. Table 2 gives the cost sensitivities associated with the optimal classic design. It allows one to know how much a small change in a single data value changes the total cost of the breakwater. This information is extremely useful during the construction process to control the cost, and for analyzing how the changes in the safety factors required by the codes influence the total cost of maritime works. For example, a change of one unit in $D_{WL}$ leads to a cost increase of 14174.2 euros (see the corresponding entry in Table 2). Similarly, an increase in the unit weight of the concrete $\gamma_c$ decreases the cost in 22562.5 euros per unit of increase.

Table 3 gives the reliability indices ($\beta$-values) sensitivities. It is useful to know how much a small change in a single data value, for example, a mean or a standard deviation, changes the corresponding $\beta$-value. In this table the designer can easily analyze how the quality of the material or precision in the construction of the work influence the safety of the breakwater. As one example, an increase of 1 unit in the friction factor $\mu_{\mu_{cr}}$, without changing the remaining data, increases the reliability index $\beta_s$ to 0.385, and an increase of the parameter $\mu_{A_u}$ of 1 unit, decreases the reliability index $\beta_o$ to 4.145.

### 5.7 Monte Carlo simulation

To determine an estimate of the global probability of failure, to understand better the interaction and correlation between modes of failure, and to estimate the probabilities of failure for each combination of modes of failure, a Monte Carlo simulation has been done with $10^9$ simulations, and the failure probabilities for each mode of failure has been determined. Table 4 shows the resulting probabilities of failure for all possible combinations of failure modes. Then, it can be concluded that the global failure probability is 0.0011, and the overtopping, sliding and armor stability probabilities of failure are $0.00025, 0.00056$ and 0.00046, respectively.

Note that these probabilities are lower than the target probabilities 0.01 and 0.001.

## 6 Conclusions

The methodology presented in this paper, denoted optimal dual safety-factor-failure-probability method, provides a rational and systematic procedure for automatic and optimal design of maritime works. The engineer is capable of observing simultaneous bounds for the safety factors and probabilities of failure against different modes of failure, so that the most stringent conditions prevail. In addition, a sensitivity analysis can be easily performed by transforming the input parameters into artificial variables, that are constrained to take their constant values. The provided example illustrates how this procedure can be applied and shows that it can be very practical and useful.

Some additional advantages of the proposed method are:

1. The reliability analysis takes full advantage of the optimization packages.
2. The Rosenblatt transformation does not need to be inverted.
3. Constraints can be written in any form, implicit or explicit.
4. The failure region need not be written in terms of the normalized (transformed) variables.
5. The responsibility for iterative methods is given to the optimization software.
6. Sensitivity values are given, without additional cost, by the values of the dual problem.
7. Monte Carlo simulation gives a better knowledge of the probabilities of occurrence of combined failures modes, and allows determining the actual global probability of failure.

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# 8 Appendix A: Cost function

Consider the rubble-mound breakwater in Figure 1. To derive the cost function the following parts are considered:

**Concrete volume:**

$$V_c = pq + se \tag{93}$$

**Armor layer volume:**

$$V_1 = ad + a\frac{b+c-t}{\sin\alpha_s} + \frac{a^2}{2\tan\alpha_s} \tag{94}$$

**Underlayer volume:**

$$V_2 = b\left(e+d-a\tan\frac{\alpha_s}{2}\right) + b\frac{c-t}{\sin\alpha_s} + \frac{b^2}{2\tan\alpha_s} + \left(2f+\frac{n+b}{\tan\alpha_\ell}\right)\frac{n+b}{2} + \frac{cb}{\sin\alpha_\ell} \tag{95}$$

**Core volume:**

$$V_3 = c\left(f + e + d - (a+b)\tan\frac{\alpha_s}{2} - \frac{b}{\sin\alpha_\ell} + \frac{n+b}{\tan\alpha_\ell}\right) + \frac{c^2}{2}\left(\frac{1}{\tan\alpha_s} + \frac{1}{\tan\alpha_\ell}\right) + t\left(r + \frac{a+b}{\sin\alpha_s}\right) \tag{96}$$

Then, the total cost becomes

$$C = C_c V_c + C_{al} V_1 + C_{ul} V_2 + C_{co} V_3. \tag{97}$$

# 9 Appendix B: Notation

$A_c$: berm level.

$A_r$: constant defining the stability function.

$A_u$: coefficient depending on the armor units.

$a$: armor layer width.

$B_r$: constant defining the stability function.

$B_u$: coefficient depending on the armor units.

$b$: underlayer width.

$C_{al}$: cost of the armor layer per unit volume.

$C_c$: cost of the concrete per unit volume.

$C_{co}$: cost of the core per unit volume.

$C_f$: experimental random coefficient.

$C_{ul}$: cost of the underlayer per unit volume.

$c$: core height.

$D_{W_L}$: design water level.

$D$: useful life.

$\mathbf{d}$: design or geometric variables.

$d$: seaside berm width.

$e$: crownwall width.

$F_a$: armor stability safety factor.

$F_a^0$: armor stability safety factor lower bound.

$F_c$: freeboard.

$F_h$: Horizontal force due to waves acting on the crownwall.

$F_i^k$: safety factor of the mode of failure $i$ at iteration $k$.

$F_o$: overtopping safety factor.

$F_o^0$: overtopping safety factor lower bound.

$F_s$: crownwall sliding safety factor.

$F_s^0$: crownwall sliding safety factor lower bound.

$F_v$: Subpressure acting on the crownwall.

$f$: leeward berm width.

$g$: Acceleration of gravity

$H$: wave height.

$H_D$: maximum wave height.

$H_s$: significant wave height.

$h_1$: astronomical tide.

$h_2$: meteorological tide.

$I_r$: Iribarren's number.

$I_{r_0}$: adimensional constant.

$L$: wave length.

$\ell$: armor equivalent cubic block side.

$\ell_e$: underlayer equivalent cubic block side.

$n$: crownwall depth.

$P_{S_0}$: wave water pressure.

$p$: upper crownwall width.

$p_i^0$: upper bound of the failure probability for mode $i$.

$q$: upper crownwall height.

$R$: adimensional constant depending on $\gamma_c$ and $\gamma_w$.

$R_u$: wave run-up.

$r$: toe width.

$S_0$: wave height due to run-up.

$s$: crownwall height.

$T$: wave period.

$T_p$: peak period.

$t$: toe height.

$tr$: tidal range.

$U$: standard uniform random variable.

$V_c$: concrete total volume.

$V_1$: armor layer total volume.

$V_2$: underlayer total volume.

$V_3$: core total volume

$v_{A_r}$: coefficient of variation of $A_r$.

$v_{B_r}$: coefficient of variation of $B_r$.

$v_{C_f}$: coefficient of variation of $C_f$.

$W$: armor block weight.

$W_f$: submerged height of the crownwall.

$W_1$: crownwall weight.

$zp$: zero port.

$\alpha$: adimensional constant.

$\alpha_\ell$: leeward slope angle.

$\alpha_s$: seaward slope angle.

$\beta_a$: armor stability reliability factor.

$\beta_o$: overtopping reliability factor.

$\beta_s$: crownwall sliding reliability factor.

$\beta_i^k$: mode $i$ reliability factor lower bound for iteration $k$.

$\beta_a^0$: armor stability reliability factor lower bound.

$\beta_o^0$: overtopping reliability factor lower bound.

$\beta_s^0$: crownwall sliding reliability factor lower bound.

$\gamma_c$: concrete unit weight.

$\gamma_s$: rubble-mound unit weight.

$\gamma_w$: water unit weight.

$\boldsymbol{\eta}$: the set of parameters used in the classic design.

$\boldsymbol{\phi}$: the set of basic random variables used only in the probabilistic design.

$\boldsymbol{\psi}$: the auxiliary (non-basic) variables which values can be obtained from those of the basic variables.

$\boldsymbol{\kappa}$: the set of parameters associated with the random variability and dependence structure of the random variables involved.

$\tilde{\boldsymbol{\eta}}$: mean value of $\boldsymbol{\eta}$.

$\tilde{\boldsymbol{\phi}}$: mean value of $\boldsymbol{\phi}$.

$\tilde{\boldsymbol{\psi}}$: mean value of $\boldsymbol{\psi}$.

$\lambda$: adimensional constant.

$\mu_{A_r}$: mean value of $A_r$.

$\mu_{B_r}$: mean value of $B_r$.

$\mu_{A_u}$: mean value of $A_u$.

$\mu_{B_u}$: mean value of $B_u$.

$\mu_{C_f}$: mean value of $C_f$.

$\mu_c$: friction factor.

$\mu_{\mu_c}$: mean value of $\mu_c$.

$\sigma_{A_u}$: standard deviation of $A_u$.

$\sigma_{B_u}$: standard deviation of $B_u$.

$\sigma_{\mu_c}$: standard deviation of $\mu_c$.

$\Phi_e$: stability function.

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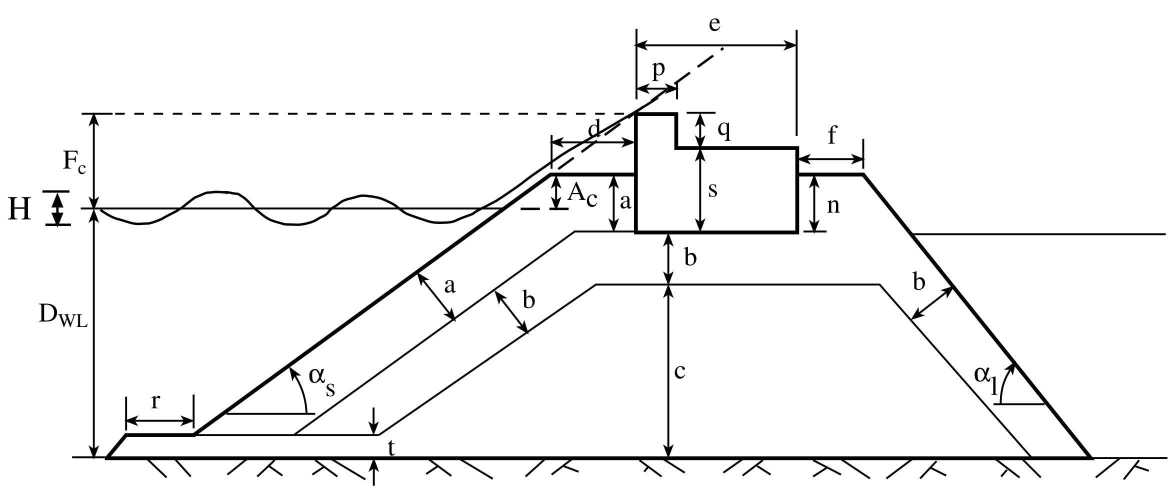


Figure 1: Parameterized rubble-mound breakwater used in the illustrative example of automatic design.

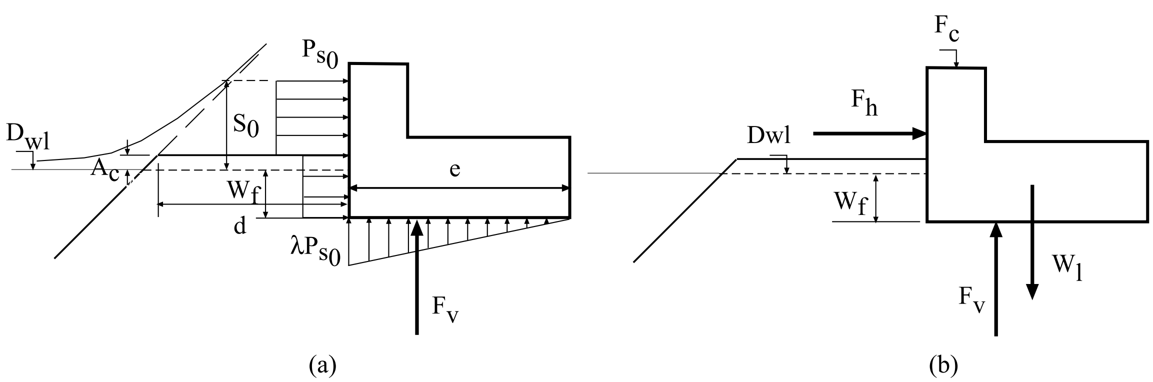


Figure 2: Illustration of the forces acting on the crownwall and producing sliding with respect to its base.

Table 1: Illustration of the iterative procedure. The design and final values are boldfaced. Active safety factors or reliability indices are underlined.

| | | ITERATIONS | | | |
|---|---|---|---|---|---|
| Variable | Units | 1 | 2 | 3 | 14 (end) |
| Cost | $euros \times 10^3$ | 272.2 | 281.9 | 291.1 | **299.7** |
| $a$ | $m$ | 6.11 | 6.18 | 6.16 | **6.16** |
| $b$ | $m$ | 2.18 | 2.20 | 2.19 | **2.19** |
| $c$ | $m$ | 16.59 | 16.50 | 17.27 | **17.47** |
| $r$ | $m$ | 2.00 | 2.00 | 2.00 | **2.00** |
| $t$ | $m$ | 1.00 | 1.00 | 1.00 | **1.00** |
| $e$ | $m$ | 10.00 | 14.51 | 16.13 | **23.54** |
| $p$ | $m$ | 2.00 | 2.00 | 2.00 | **2.00** |
| $s$ | $m$ | 6.61 | 12.00 | 12.00 | **12.00** |
| $q$ | $m$ | 7.13 | 8.00 | 8.00 | **8.00** |
| $d$ | $m$ | 6.11 | 6.18 | 6.16 | **6.16** |
| $f$ | $m$ | 2.18 | 2.20 | 2.19 | **2.19** |
| $n$ | $m$ | 0.00 | 0.00 | 0.00 | **0.00** |
| $\beta$ | $m$ | 0.46 | 0.46 | 0.46 | **0.46** |
| $\alpha$ | $m$ | 0.59 | 0.59 | 0.59 | **0.59** |
| $F_c$ | $m$ | 12.50 | 18.70 | 19.46 | **19.67** |
| $F_o$ | – | 1.21 | 1.81 | 1.88 | **1.90** |
| $F_s$ | – | $\underline{1.50}$ | 3.92 | 7.41 | **12.58** |
| $F_a$ | – | $\underline{1.50}$ | 1.55 | 1.53 | **1.54** |
| $\beta_o$ | – | 2.70 | 2.88 | 3.23 | **3.32** |
| $\beta_s$ | – | 1.87 | 1.33 | 2.27 | $\underline{\mathbf{3.08}}$ |
| $\beta_a$ | – | 2.92 | 3.13 | 3.07 | $\underline{\mathbf{3.08}}$ |

Table 2: Cost sensitivities $\frac{\partial C}{\partial x}$ with respect to the data values in the crownwall illustrative example. Positive values indicate a cost increase when the corresponding parameter is increased one unit.

| $\partial C/\partial x$ | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $D_{WL}$ | $\gamma_c$ | $\gamma_s$ | $\gamma_w$ | $F_o$ | $F_s$ |
| | 14174.2 | -22562.5 | -47.3 | 51848.6 | 0.000 | 1599.5 |
| $x$ | $F_a$ | $c_{al}$ | $c_{ul}$ | $c_{co}$ | $c_c$ | $H_s$ |
| | 62258.2 | 335.9 | 226.0 | 1070.2 | 298.5 | 40391.0 |

Table 3: Cost sensitivities $\frac{\partial C}{\partial x}$ with respect to the data values in the crownwall illustrative example. Positive values indicate a cost increase when the corresponding parameter is increased one unit.

| Data value $x$ | $\frac{\partial \beta_o}{\partial x}$ | $\frac{\partial \beta_s}{\partial x}$ | $\frac{\partial \beta_a}{\partial x}$ |
|---|---|---|---|
| $\gamma_c$ | 0.000 | 0.021 | 2.198 |
| $\gamma_s$ | 0.000 | 0.000 | 0.000 |
| $\gamma_w$ | 0.000 | -0.048 | -6.204 |
| $\mu_{A_u}$ | -4.145 | -0.708 | 0.000 |
| $\mu_{B_u}$ | 2.239 | 0.431 | 0.000 |
| $\mu_{\mu_{cr}}$ | 0.000 | 0.385 | 0.000 |
| $\mu_{A_r}$ | 0.000 | 0.000 | -308.402 |
| $\mu_{B_r}$ | 0.000 | 0.000 | -0.158 |
| $\mu_{c_f}$ | 0.000 | -0.336 | 0.000 |
| $\mu_{h_1}$ | -0.028 | -0.167 | -0.258 |
| $\mu_{h_2}$ | -0.028 | -0.167 | -0.258 |
| $\sigma_{A_u}$ | -32.994 | -3.343 | 0.000 |
| $\sigma_{B_u}$ | -6.032 | -0.783 | 0.000 |
| $\sigma_{\mu_{cr}}$ | 0.000 | -0.091 | 0.000 |
| $\upsilon_{A_r}$ | 0.000 | 0.000 | -47.323 |
| $\upsilon_{B_r}$ | 0.000 | 0.000 | -23.132 |
| $\upsilon_{C_f}$ | 0.000 | -0.587 | 0.000 |
| $\sigma_{h_1}$ | -0.019 | -0.162 | -0.245 |
| $\sigma_{h_2}$ | -0.001 | -0.104 | -0.114 |

Table 4: Monte Carlo estimation of the probabilities of failure for all combinations of failure modes.

| Type of failure | Probability |
|---|---|
| $\emptyset$ | 0.99890 |
| $\{o\}$ | 0.00010 |
| $\{s\}$ | 0.00039 |
| $\{a\}$ | 0.00044 |
| $\{o\} \cup \{s\}$ | 0.00015 |
| $\{o\} \cup \{a\}$ | 0.00000 |
| $\{s\} \cup \{a\}$ | 0.00002 |
| $\{o\} \cup \{s\} \cup \{a\}$ | 0.00000 |

## Figure captions

Figure 1. Parameterized rubble-mound breakwater used in the illustrative example of automatic design.

Figure 2. Illustration of the forces acting on the crownwall and producing sliding with respect to its base.

## Table captions

Table 1. Illustration of the iterative procedure. The design and final values are boldfaced. Active safety factors or reliability indices are underlined.

Table 2. Cost sensitivities $\dfrac{\partial C}{\partial x}$ with respect to the data values in the crownwall illustrative example. Positive values indicate a cost increase when the corresponding parameter is increased one unit.

Table 3. Reliability index sensitivities $\frac{\partial \beta_i}{\partial x}$, with respect to the data values $x$, in the illustrative example. Positive values indicate an index increase when the corresponding parameter is increased one unit.

Table 4. Monte Carlo estimation of the probabilities of failure for all combinations of failure modes.