Computing Failure Probabilities. Applications to Reliability Analysis

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Abstract

The paper presents one method for calculating failure probabilities with applications to reliability analysis. The method is based on transforming the initial set of variables to a n-dimensional uniform random variable in the unit hypercube, together with the limit condition set and calculating the associated probability using a recursive method based on the Gauss-Legendre quadrature formulas to calculate the resulting multiple integrals. An example of application is used to illustrate the proposed method.

Key Words: Extreme percentiles, Gauss-Legendre quadrature, Recursive method, Tail approximation.

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Abstract

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1 Introduction

Since the pioneering works of Freudenthal [13] in the fifties, safety analyses have been based on probabilistic concepts and the computation of the probability of failure, in a more or less direct fashion (see Ditlevsen [8]). Consequently, the classical methods, based on partial safety coefficients, are in the process of being abandoned.

In the reliability of a system, there are many variables $(X_1, X_2, \ldots, X_n)$ involved. They belong to an n-dimensional space, which can be divided in two regions (see Melchers [22]): the safe and the failure regions (see Figure 1).

$$\begin{array}{ll} \text{Safe Region:} & W = g(x_1, x_2, \ldots, x_n) > 0, \\ \text{Failure Region:} & W = g(x_1, x_2, \ldots, x_n) \leq 0, \end{array} \tag{1}$$

where $W = g(X_1, X_2, \ldots, X_n)$ is a random variable. The boundary of such a regions is defined by the system limit states.

Since Equation (1) is a limit condition, and some extra safety is needed, the design of a system is done in the safe region using the condition

$$W = g(x_1, x_2, \ldots, x_n) = w_0 > 0$$

where w_0 is the security margin, sometimes known as the *failure indicator*. Designing a sufficiently safe solution means obtaining an adequate value of w_0 (the design value of W), associated with a sufficiently small failure probability. However, the computation of w_0 is not an easy task, and some simplifications are required. This, leads to the main three methods used in reliability analysis:

1. Level 1: To select the design value w_0, partial safety coefficients related to some design variables (loads, strengths, etc.) are used. It is the classical method.

   To illustrate we represent in Figure 1 the case of two variables X_1 and X_2. Assume that X_1 and X_2 are two design variables, that become more dangerous as they decrease and increase their values, respectively. Then, a code based on partial safety coefficients fixes two values x_1 and x_2, and states that such a set of values (x_1, x_2) is safe if and only if the point $(x_1/\beta_1, \alpha_2 x_2)$ is in the safe region, where α_1 and β_2 are the partial safety coefficients for both variables. Other combinations can arise for other pairs of variables (points $(\alpha_1 x_1, x_2/\beta_2)$, $(\alpha_1 x_1, \alpha_2 x_2)$ and $(x_1/\beta_1, x_2/\beta_2)$) (see Figure 1), depending on whether they become more or less dangerous when they increase their values. Thus, the design value w_0 depends on the selected partial security factors.

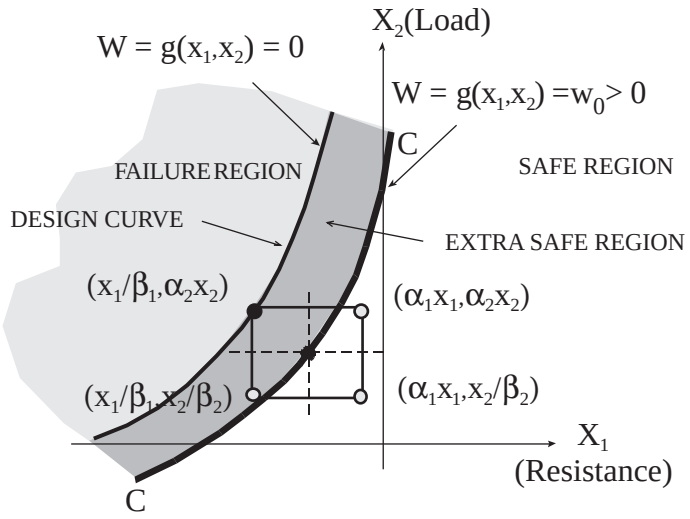

Figure 1: Illustration of the partial safety coefficients.

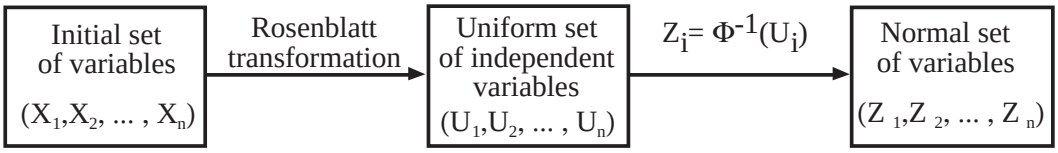

Figure 2: Illustration of how the initial set of variables is transformed, first into a set of independent uniform random variables, and later to a set of independent normal random variables.

2. Level 2: Alternatively, we can select the value of w_0 by fixing a probability of failure, P_f, that can be calculated using the joint probability density $f(\mathbf{x}) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ of all variables involved, by means of the integral:

$$P_f = P(W \leq 0) = F_W(0) = \int\limits_{g(x_1,x_2,\ldots,x_n)\leq 0} f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)dx_1dx_2 \ldots dx_n. \qquad (2)$$

where $F_W(\cdot)$ is the cumulative distribution function of W.

The problem is that the integral (2) is usually difficult to calculate, due to two main reasons: (a) the complicated expression of the density $f(\mathbf{x})$, and (b) the complicated form of the region $g(\mathbf{x}) \leq 0$. Thus, approximate methods are used to calculate the failure probability, that are based on some approximations of the integration region $g(x_1, x_2, \ldots, x_n) \leq 0$, the density function $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$, or both. At this level, they are normally based on an approximate representation of the probability distribution using first and second order moments of the joint distribution. This method is exact when normal variables are used, because first and second order moments characterize these variables.

3. Level 3: The design value w_0 is computed using the full representation of the joint distribution and/or the exact failure region appearing in Expression (2). Thus, methods at this level require especial integration formulas and methods.

To approximate the integral in (2) the initial set of variables can be transformed into a set of independent standard unit normal variables, as illustrated in Figure 2, and then two main approaches can be used: First and second order reliability models.

The 'First Order Reliability Methods", that use linear approximations, appeared in the field of structural reliability with Freudenthal [13] in 1956, and has been expanded by Hasofer and Lind [18], Rackwitz and Flessler [24], Breitung [1], etc.

Second order methods consider some partial Taylor series expansions to approximate the functions or regions involved (see, for example, Davies [7], Field [12], Breitung [1], Tvedt [29], Katsuki and Frangopol [20], or Papadimitriou [23]). For a complete description of some of these methods and some illustrative examples see Ditlevsen and Madsen [10] and Madsen, Krenk and Lind [21]. These methods have shown to give precise results and have demonstrated to be much more efficient than Monte Carlo simulation techniques for estimating extreme percentiles (see, for example, Wirsching and Wu [30], or Haskin, Staple and Ding [16]).

One well-known alternative technique to deal with this problem is importance or weighted sampling (see for example, Siegmund [28], Rubinstein [27], Ripley [25], Zhang [32], or Givens and Raftery [15]). Importance sampling techniques can be used to decrease simulation time required for rare event simulations and reduce variance (see Hesterberg [19]). In fact, for approximating tail probabilities methods such as importance sampling are needed to get acceptable approximations.

The reader must be aware of the fact that methods for calculating the probability of failure present problems when this probability is small, as it is commonly the case in reliability analysis. In this case we say that we are estimating tail (very small) probabilities that are well known to be very sensitive to the assumed model (see Ditlevsen [9], Galambos [14] and Castillo [3]). In other words, the failure probability is strongly dependent on the tail assumptions. Since, in reliability analysis we are normally dealing with very small probabilities, adequate simulation or calculation methods (direct Monte Carlo methods are very inefficient) are needed. Some of them can be seen in Castillo, Solares and Gómez [4, 5, 6], Wirsching and Wu [30] and Wu, Burnside and Cruse [31]).

Thus, it is important to use integration methods that take into account the exact form of the integration region, and that are independent on the size of it. On the other hand, transformations of the random variables involved must be reduced to a minimum and the final n-dimensional space must be simple and, if possible, finite. In this paper we present one level 3 method to calculate the probability of failure that uses Expression (2), and satisfies these requirements.

The paper is structured as follows. In Section 2 we introduce the proposed method and give a detailed description of it. In Section 3 we analyze and discuss the performance of this method. In Section 4 we give one example of application where all its main steps can be easily understood. Finally, in Section 5 we give some conclusions and recommendations.

2 The proposed method

In this paper we deal with level 3 and propose one method for calculating the probability of failure P_f, i.e., the integral in (2), that uses the Gauss-Legendre quadrature formulas for the n-dimensional region.

The process consists of four main steps:

Step 1: Transforming the variables to the unit hypercube In this step, the initial set of variables are transformed to a set of variables with standard uniform marginals $U(0, 1)$.

Step 2: Forcing the origin to belong to the failure region. This simplifies the identification of the failure region.

Step 3: Characterizing the failure region. This allows calculating the n-dimensional integral.

Step 4: Evaluating the integral. Using the Gauss-Legendre quadrature formulas in a recursive algorithm.

2.1 Transforming the variables to the unit hypercube.

In this subsection we describe two methods for obtaining new variables with uniform marginals.

Figure 3: Illustration of how the initial set of variables is transformed into a set of possibly dependent uniform random variables.

2.1.1 Rosenblatt Transformation

At this point, it is interesting to ask whether or not it is possible to transform a given n-dimensional random variable to a n-dimensional set of independent uniform variables. The answer to this question is positive and the required transformation can be obtained using an interesting result due to Rosenblatt [26]. This transformation is:

$$\begin{array}{rcl} U_1 &=& F_1(X_1) \\ U_2 &=& F_2(X_2|X_1) \\ \vdots & \vdots & \vdots \\ U_n &=& F_n(X_n|X_1, X_2, \ldots, X_n), \end{array} \tag{3}$$

where $F_1(X_1)$, $F_2(X_2|X_1), \ldots, F_n(X_n|X_1, X_2, \ldots, X_n)$ are the cdf of the X_1 marginal and the cdfs of the indicated conditional random variables.

The main advantage of this transformation is that the function $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ in (2) reduces to the unit constant, and then, the integral in (2) becomes the volume of the failure region.

In addition, it is worthy mentioning that the Rosenblatt transformation leads to non-dimensional random variables. Note that the cdf functions in (3) are probabilities, and then they have no dimensions (they are invariant with respect to changes in the units of measure used for the random variables involved).

2.1.2 The cdf marginal transformation

An alternative consists of performing the transformation (see Figure 3)

$$U_i = F_{X_i}(X_i);\ i = 1, 2, \ldots, n. \tag{4}$$

Then, if $f_{X_1,X_2,\ldots,X_n)}(x_1, x_2, \ldots, x_n)$ is the pdf of the initial set of random variables $X_1, X_2, \ldots, X_n$, the pdf of the new variables $U_1, U_2, \ldots, U_n$ becomes

$$g_{U_1,U_2,\ldots,U_n}(u_1, u_2, \ldots, u_n) = f_{X_1,X_2,\ldots,X_n}(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2), \ldots, F_{X_n}^{-1}(u_n)) \prod_{i=1}^{n} \frac{1}{f_{X_i}(F_{X_i}^{-1}(u_i))} \tag{5}$$

This alternative also reduces the n-dimensional space to the finite unit hypercube. However, neither the transformed integration function $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ nor the integration region $g(x_1, x_2, \ldots, x_n) \leq 0$ are simplified.

It is worthwhile mentioning that with this solution we do not miss the possible dependencies among the variables, because a one to one transformation is used. The reader should not confuse this approach with other approaches that use joint distributions with the same marginals as the original distribution, as for example, the product of all marginals, that would imply loosing the dependence properties of the original random variables.

2.2 Forcing the origin to belong to the failure region

In this section, we assume that the random variables involved are such that any change in the value of a single variable has a monotone influence on the safety of the system being studied. If this is not the case,

such a variable can either be removed, or it behaves in an strange form. This assumption implies that either the end point value 0 or the end point value 1 for such variable is the worst possible value with respect to the safety of the system. Then, it is clear that if we transform the variables U_k with 1 as their worst possible values, to $1 - U_k$, we get new variables such that the origin is the worst possible combination of values for all variables, and then, the origin belongs to the failure region.

2.3 Characterizing the failure region

Our problem consists of approximating as well as possible the integral

$$P_f = P(W \leq 0) = F_W(0) = \int\limits_{g(u_1,u_2,\ldots,u_n)\leq 0} f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n)du_n du_{n-1}\ldots du_1. \tag{6}$$

that can be written as

$$P_f = \int_0^{b_1}\int_0^{b_2(u_1)}\int_0^{b_3(u_1,u_2)}\ldots\int_0^{b_n(u_1,u_2,\ldots,u_{n-1})} f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n)du_n du_{n-1}\ldots du_1. \tag{7}$$

So, expressions for b_1 and the functions $b_2(u_1), b_3(u_1,u_2),\ldots,b_n(u_1,u_2,\ldots,u_{n-1})$, are required. To this end, we consider the inverse function, g_i^{-1}, with respect to its ith argument, that is defined by the identity

$$g(u_1,u_2,\ldots,u_n) = 0 \Leftrightarrow g_i^{-1}(u_1,u_2,\ldots,u_{i-1},0,u_{i+1},u_{i+2},\ldots,u_n) = u_i. \tag{8}$$

Taking into account that the uniform random variables have 0 and 1 as lower and upper ends, respectively, and the origin is in the failure region, we can write

$$b_k(u_1,u_2,\ldots,u_{k-1}) = \min\left(g_k^{-1}(u_1,u_2,\ldots,u_{k-1},0,\ldots,0),1\right) \tag{9}$$

where

$$b_1 = min\left(g_1^{-1}(0,\ldots,0),1\right). \tag{10}$$

These expressions are based on the monotone influence of the random variables on the safety.

Since the inverse functions above could not be obtained analytically, numerical methods are usually required. Among them, due to the monotone and bounded (with bounds 0 and 1) character of the involved variables, the bisection method becomes especially convenient. The main reason for using the bisection method is only that we look for a method without convergence problems for any possible failure region. Other methods are generally faster, but can fail with some functions.

2.4 Evaluation of the integral using the Gauss-Legendre quadrature formulas

This variant consists of using the Gauss-Legendre approximation

$$\int_a^b f(u)dx \simeq \frac{b-a}{2}\sum_{i=0}^{m} w_i f\left(\frac{z_i(b-a)+b+a}{2}\right), \tag{11}$$

where w_i and z_i are the weights and the corresponding selected points of the Gauss-Legendre integration method (see Carnahan, Luther and Wilkes [2]).

To implement this one dimensional formula for the n-dimensional integral (7), we write

$$I_k = Integral(u_1,u_2,\ldots,u_{k-1}) = \int_0^{b_k(u_1,\ldots,u_{k-1})}\ldots\int_0^{b_n(u_1,u_2,\ldots,u_{n-1})} f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n)dx_n\ldots dx_k \tag{12}$$

with

$$Integral(u_1,u_2,\ldots,u_n) = f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n), \tag{13}$$

which can be approximated by

$$\begin{aligned} I_k = Integral(u_1,u_2,\ldots,u_{k-1}) &\simeq \frac{b_k(u_1,u_2,\ldots,u_{k-1})}{2} \\ &\times \sum_{i=0}^{m} w_i Integral\left(u_1,u_2,\ldots,u_{k-1},\frac{z_i b_k(u_1,u_2,\ldots,u_{k-1})+b_k(u_1,u_2,\ldots,u_{k-1})}{2}\right), \end{aligned} \tag{14}$$

which allows us calculating $P_f = I_1$ using a recursion.

2.5 Algorithm

Thus, the proposed method can be applied using the following algorithm.

Algorithm 1 Calculating the probability of failure

- **Input:** The initial set of variables $\{X_1, X_2, \ldots, X_n\}$, its pdf $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$, and the failure region $g_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) \leq 0$.
- **Output:** The probability of failure (associated with the given failure region).

Step 1: Use the Rosenblatt transformation or the cdf marginal transformation to transform the initial set of variables $\mathbf{X}$ to a set of uniform $U(0,1)$ random variables $\mathbf{V}$, and the failure function $g(\mathbf{x})$ to the transformed failure function $g_V(\mathbf{v})$.

Step 2: Transform the variables $\mathbf{V}$ such that the vector $(0, 0, , \ldots, 0)$ becomes the worst possible vector. To this end, change V_i to $U_i = V_i$, or $U_i = 1 - V_i$, for the necessary i values.

Step 3: Obtain the transformed failure region $g_{U_1,\ldots,U_n}(u_1, \ldots, u_n)$, and the pdf $f_{U_1,\ldots,U_n}(u_1, \ldots, u_n)$ of the new random variables (it will be the unit constant if the Rosenblatt transformation was used in Step 1).

Step 4: Obtain the probability of failure using the recursion (14), based on the Gauss-Legendre quadrature formula (11).

3 Performance of the method

To test the behavior of the method we have assumed that the failure region is

$$\sum_{i=1}^{n} X_i \leq n/3 - 1,$$

i.e., the sum of n uniform random variables $U(0,1)$. This allows us checking the results, because we know its exact cdf:

$$F(x) = \frac{1}{n!} \sum_{r=0}^{\lfloor x \rfloor} (-1)^r \left(\begin{array}{c} n \\ r \end{array} \right) (x - r)^n; \;\; 0 < x < n, \tag{15}$$

where $\lfloor x \rfloor$ is the integer part of x.

To save time in the calculations we have used a crude version and an optimized version in which the degree q of the polynomial of the Gauss approximation is a function of the length b of the interval where the unidimensional integrations are calculated, using the following criterion

$$q = m(1 + b)/2.$$

Note that the degree of the polynomial ranges from a minimum of $m/2$, for $b = 0$, to a maximum of m, for $b = 1$.

Table 1 shows an illustration of the quality of the approximation as a function of the degree m of the Gauss approximation and the number of variables involved, for the optimized and non-optimized procedures for the sum of n uniform random variables $U(0,1)$.

A look at Table 1 leads to the following conclusions:

1. The savings, in the number of function calls and the cpu time, obtained using the optimized method are important.
2. The quality of the approximation is not very much influenced by the optimization procedure.

Table 1: Illustration of the quality of the approximation as a function of the degree of the Gauss approximation and the number of variables involved, for the optimized and non-optimized procedures.

		Optimized			Non-Optimized	
n	m	Estimated probability	cpu time (in ticks)	Exact probability	Estimated probability	cpu time (in ticks)
5	2	0.000129	0	0.001097	0.001067	0
5	3	0.001058	0	0.001097	0.001097	0
5	4	0.001097	0	0.001097	0.001097	0
8	2	0.000451	0	0.001469	0.001271	0
8	3	0.001428	0	0.001469	0.001471	3
8	4	0.001468	2	0.001469	0.001469	26
10	2	0.000652	0	0.001269	0.001096	1
10	3	0.001244	1	0.001269	0.001272	32
10	4	0.001269	23	0.001269	0.001269	438
12	2	0.000523	0	0.001007	0.000869	4
12	3	0.000998	8	0.001007	0.001010	307
12	4	0.001006	280	0.001007	0.001007	7368
15	2	0.000391	1	0.000659	0.000563	36
15	3	0.000650	136	0.000659	0.000659	9016
15	4	0.000658	12843	0.000659	0.000659	-

3. In the case of the sums of uniforms the degree of the polynomial required for obtaining a good approximation to the exact value of the probability of failure is very small. In fact, a polynomial of degree four gives almost the exact values.

4. The number of function calls and the cpu time required for the calculations increases exponentially with the number of random variables involved.

4 Example of Application

In this section we consider a practical application and we make two assumptions for the random distributions: **Case 1**: independent lognormal distributions to allow for the exact probability of failure to be calculated, and **Case 2**: independent normal distributions.

4.1 Statement of the problem

Consider the simply supported beam in Figure 4, where P is a point load applied in the middle of the beam, L is the beam length, and b and c are the dimensions of its rectangular cross section.

To analyze the beam problem we consider the following initial set of variables:

- **Random variables:**
  - P: Maximum value of the point load that occurs during the service life of the structure.
  - s: Actual beam strength.
  - L: The actual beam length.
  - b: The actual width depth.
  - c: The actual beam depth.
- **Design variables:**

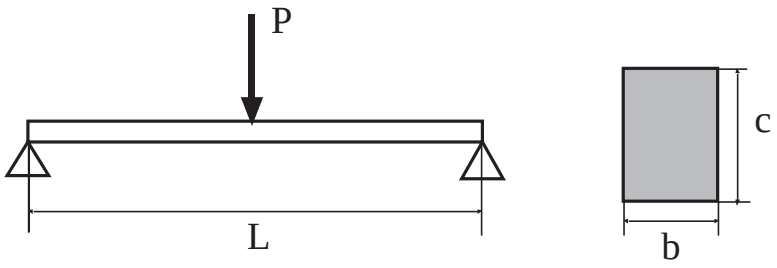

Figure 4: A simply supported beam and its cross section.

- P_0: Point load fixed by the code for this type of structure.
- s_0: Beam strength fixed by the code.
- L_0: The design beam length.
- b_0: The design beam width.
- c_0: The design beam depth.
- γ_P: Safety factor used to majorize the load P.
- γ_s: Safety factor used to minorize the strength s_0.

We note that we have distinguished the design variables L_0, b_0 and c_0, which are fixed values, from the actual values L, b and c, which are random variables (normally assumed to be with mean values L_0, b_0 and c_0, respectively, and small variance).

4.2 Classical design

In this section we use a classical design, i.e., based on safety coefficients. Then, the geometric design of the beam, i.e., the determination of the values of b_0 and c_0, can be done using the well known strength of materials expression:

$$\frac{s_0}{\gamma_s} = \frac{M}{W} = \frac{\gamma_P P_0 L_0/4}{\frac{I_0}{c_0/2}} = \frac{3\gamma_P P_0 L_0}{2b_0 c_0^2}, \tag{16}$$

where W_0 and I_0 are the resistant moment and the moment of inertia, respectively, associated with b_0 and c_0.

4.3 Failure region associated with the example of application

Since the random variable maximum stress s_{max} is related to the random variables by the expression

$$s_{max} = \frac{3PL}{2bc^2}, \tag{17}$$

the system will fail if $s_{max} \geq s$. Then, the failure region will be:

$$\frac{3PL}{2bc^2} \geq s, \tag{18}$$

i.e.,

$$\begin{array}{lrcl} \text{Safe Region:} & s - (3PL)/(2bc^2) & > & 0, \\ \text{Failure Region:} & s - (3PL)/(2bc^2) & \leq & 0. \end{array} \tag{19}$$

4.4 Case 1: Lognormal distributions

4.4.1 Assumptions for the random variables involved

For the sake of simplicity and with the aim of illustrating the design process and being able to calculate the exact probability of failure, we assume in this case that P, s, L, b and c are independent lognormal distributions:

$$\begin{aligned} \log P &\sim N(\mu_P, \sigma_P^2), \\ \log s &\sim N(\mu_s, \sigma_s^2), \\ \log L &\sim N(\mu_{L_0}, \sigma_L^2), \\ \log b &\sim N(\mu_{b_0}, \sigma_b^2), \\ \log c &\sim N(\mu_{c_0}, \sigma_c^2). \end{aligned}$$

4.4.2 Exact failure probability associated with a classical design

The exact probability of failure associated with this classical design can be obtained using the properties of the lognormal distributions, as follows.

$$\begin{aligned} p_F &= Prob\left[s_{max} \geq s\right] = Prob\left[\frac{3PL}{2bc^2} \geq s\right] = \\ &= Prob\left[-\log P + \log s - \log L + \log b + 2\log c + \log\frac{2}{3} \leq 0\right] = \\ &= \underset{N(-\mu_P + \mu_s - \mu_L + \mu_b + 2\mu_c + \log\frac{2}{3},\, \sigma_P^2 + \sigma_s^2 + \sigma_L^2 + \sigma_b^2 + 4\sigma_c^2)}{F}^{(0)} \\ &= \Phi\left[\frac{-\log\frac{2}{3} + \mu_P - \mu_s + \mu_L - \mu_b - 2\mu_c}{\sqrt{\sigma_P^2 + \sigma_s^2 + \sigma_L^2 + \sigma_b^2 + 4\sigma_c^2}}\right], \end{aligned} \tag{20}$$

where Φ is the cdf of the standard normal $N(0,1)$, random variable.

4.4.3 Level 2: FORM and SORM approximations

FORM methods transform the initial set of random variables to a set $\mathbf{Z}$ of standard independent $N(0,1)$ random variables

$$Z_1 = \frac{\log P - \mu_P}{\sigma_P}; \quad Z_2 = \frac{\log s - \mu_s}{\sigma_s}; \quad Z_3 = \frac{\log L - \mu_L}{\sigma_L}; \quad Z_4 = \frac{\log b - \mu_b}{\sigma_b}; \quad Z_5 = \frac{\log c - \mu_c}{\sigma_c}, \tag{21}$$

Then, taking logarithms in the failure region (18) and replacing (21), the failure region becomes

$$g_Z(\mathbf{Z}) = -\mu_P - \sigma_P Z_1 + \mu_s + \sigma_s Z_2 - \mu_L - \sigma_L Z_3 + \mu_b + \sigma_b Z_4 + 2\mu_c + 2\sigma_c Z_5 + \log(2/3) \leq 0, \tag{22}$$

Since (22) is already linear in the Z variables, then SORM and FORM designs coincide.
From (22) we get

$$P_f = \Phi\left(\frac{\mu_P - \mu_s + \mu_L - \mu_b - 2\mu_c - \log(2/3)}{\sqrt{\sigma_P^2 + \sigma_s^2 + \sigma_L^2 + \sigma_b^2 + 4\sigma_c^2}}\right). \tag{23}$$

4.4.4 Level 3: Proposed method

In this section we determine the probability of failure using the proposed method and algorithm. Thus, we proceed as follows:

Step 1: We use the Rosenblatt transformation to transform the initial set of variables into a set of independent uniform $U(0,1)$ random variables V. To this end, we let:

$$\begin{array}{lll} \log P = & X_1 = & F^{-1}_{N(\mu_P,\sigma_P^2)}(V_1) = \mu_P + \sigma_P \Phi^{-1}(V_1) \\ \log s = & X_2 = & F^{-1}_{N(\mu_s,\sigma_s^2)}(V_2) = \mu_s + \sigma_s \Phi^{-1}(V_2) \\ \log L = & X_3 = & F^{-1}_{N(\mu_L,\sigma_L^2)}(V_3) = \mu_L + \sigma_L \Phi^{-1}(V_3) \\ \log b = & X_4 = & F^{-1}_{N(\mu_b,\sigma_b^2)}(V_4) = \mu_b + \sigma_b \Phi^{-1}(V_4) \\ \log c = & X_5 = & F^{-1}_{N(\mu_c,\sigma_c^2)}(V_5) = \mu_c + \sigma_c \Phi^{-1}(V_5) \end{array} \tag{24}$$

that is,

$$\begin{array}{l} V_1 = F_{(\mu_P,\sigma_P^2)}(X_1) = \Phi\left(\dfrac{X_1-\mu_P}{\sigma_P}\right) \\ V_2 = F_{(\mu_s,\sigma_s^2)}(X_2) = \Phi\left(\dfrac{X_2-\mu_s}{\sigma_s}\right) \\ V_3 = F_{(\mu_L,\sigma_L^2)}(X_3) = \Phi\left(\dfrac{X_3-\mu_L}{\sigma_L}\right) \\ V_4 = F_{(\mu_b,\sigma_b^2)}(X_4) = \Phi\left(\dfrac{X_4-\mu_b}{\sigma_b}\right) \\ V_5 = F_{(\mu_c,\sigma_c^2)}(X_5) = \Phi\left(\dfrac{X_5-\mu_c}{\sigma_c}\right) \end{array} \tag{25}$$

And the transformed failure region in terms of the V random variables becomes

$$-\mu_P - \sigma_P \Phi^{-1}(V_1) + \mu_s + \sigma_s \Phi^{-1}(V_2) - \mu_L - \sigma_L \Phi^{-1}(V_3) + \mu_b + \sigma_b \Phi^{-1}(V_4) + 2\mu_c + 2\sigma_c \Phi^{-1}(V_5) + \log\frac{2}{3} \leq 0. \tag{26}$$

Step 2: We transform the variables $\mathbf{V}$ such that the vector $(0,0,,\ldots,0)$ becomes the worst possible vector. To this end, we let $U_1 = 1 - V_1$ and $U_3 = 1 - V_3$. Thus, we get the transformation:

$$\begin{array}{lll} U_1 & = & 1 - V_1 \\ U_2 & = & V_2 \\ U_3 & = & 1 - V_3 \\ U_4 & = & V_4 \\ U_5 & = & V_5 \end{array} \tag{27}$$

Step 3: Since the new variables U are iid uniform $U(0,1)$, we get

$$f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n) = 1;\ 0 \leq u_1 \leq 1;\ i = 1,2,3,4,5,$$

and the transformed failure region becomes

$$-\mu_P - \sigma_P \Phi^{-1}(1-U_1) + \mu_s + \sigma_s \Phi^{-1}(U_2) - \mu_L - \sigma_L \Phi^{-1}(1-U_3) + \mu_b + \sigma_b \Phi^{-1}(U_4) + 2\mu_c + 2\sigma_c \Phi^{-1}(U_5) + \log\frac{2}{3} \leq 0. \tag{28}$$

Step 4: We calculate the probability of failure using the Gauss-Legendre recursion Expression (14).

4.4.5 Numerical example

In this numerical example we assume that P, s, L, b and c are independent random variables such that the means and standard deviations for their logarithms are shown in Table 2.

4.4.6 Classical design

Assume that the values given by the code are:

$$s_0 = 25Mp; \;\; P_0 = 400000N; \;\; L_0 = 10m; \;\; \gamma_P = 1.6; \;\; \gamma_s = 1.5. \tag{29}$$

Then, according to Expression (16) we get

$$b_0 c_0^2 = 0.576. \tag{30}$$

So, we are free to choose any pair of values for variables b and c such that (30) holds. For example, we can design using $b_0 = 0.576$m and $c_0 = 1.0$m.

According to Expression (20) the probability of failure associated with the classical design is

$$P_f = \Phi(-\beta) = \Phi(-3.19785) = 0.000692289.$$

4.4.7 Level 2: FORM and SORM probabilities of failure

The FORM failure probability is based on Equation (23):

$$P_f = \Phi\left(\frac{\mu_P - \mu_s + \mu_L - \mu_b - 2\mu_c - \log(2/3)}{\sqrt{\sigma_P^2 + \sigma_s^2 + \sigma_L^2 + \sigma_b^2 + 4\sigma_c^2}}\right) \tag{31}$$

where $\mu_b = b_0$ and $\mu_c = c_0$ are the selected values that satisfies (30).

The design point P^*, such that its distance to the origin, in the transformed domain (the one associated with standard normal variables) is a minimum, can be calculated using the bisection method because we know the direction of the vector $\vec{OP^*}$. Thus, we get

$$P^* = (Z_1^*, Z_2^*, Z_3^*, Z_4^*, Z_5^*) = (3.01348, -0.994881, 0.268962, -0.128874, -0.257749),$$

which leads to the design values(see (21))

$$\begin{array}{rcccl} P^* & = & e^{\mu_P + Z_1^* \sigma_P} & = & 870349N \\ s^* & = & e^{\mu_s + Z_2^* \sigma_s} & = & 22.97Mp \\ L^* & = & e^{\mu_L + Z_3^* \sigma_L} & = & 10.06m \\ b^* & = & e^{\mu_b + Z_4^* \sigma_b} & = & 0.575m \\ c^* & = & e^{\mu_c + Z_5^* \sigma_c} & = & 0.997m \end{array}$$

and to the following partial security coefficients

$$\begin{array}{rcccl} \gamma_p & = & e^{\mu_P + P^* \sigma_P} / e^{\mu_p} & = & 2.176 \\ \gamma_s & = & e^{\mu_s} / e^{\mu_s + s^* \sigma_s} & = & 1.0884 \\ \gamma_L & = & e^{\mu_L + L^* \sigma_L} / e^{\mu_L} & = & 1.0062 \\ \gamma_b & = & e^{\mu_b} / e^{\mu_b + b^* \sigma_b} & = & 1.0014 \\ \gamma_c & = & e^{\mu_c} / e^{\mu_c + c^* \sigma_c} & = & 1.0028 \end{array} \tag{32}$$

Table 2: Means and standard deviations of the logarithms of the 5 random variables involved in the practical example 1.

Variable	Mean	Standard deviation
P	$\mu_P = \log(400000) = 12.8992$	$\sigma_P = 0.257984$
s	$\mu_s = \log(25000000) = 17.0344$	$\sigma_s = 0.085172$
L	$\mu_L = \log(10) = 2.30259$	$\sigma_L = 0.023026$
b	$\mu_b = \log(0.576) = -0.551648$	$\sigma_b = 0.011033$
c	$\mu_c = \log(1) = 0$	$\sigma_c = 0.011033$

Table 3: Different estimated values of the probability of failure for different approximations of the Gauss Legendre quadrature using the optimized method.

n	m	Estimated probability	Exact probability	cpu time (in ticks)
5	5	0.000000	0.000696	1
5	10	0.000001	0.000696	11
5	20	0.000307	0.000696	250
5	30	0.000555	0.000696	1649
5	40	0.000657	0.000696	6834
5	50	0.000692	0.000696	21896

and a global safety factor value of $\gamma_G = \gamma_P\gamma_s\gamma_L\gamma_b\gamma_c = 2.393$.

Note that the partial safety coefficients, γ_L, γ_b and γ_c, for L, b and c, respectively, are small, as expected because their assumed standard deviations are small.

The associated probability of failure becomes:

$$P_f = \Phi(-\beta) = \Phi\left(-\sqrt{Z_1^{*2} + Z_2^{*2} + Z_3^{*2} + Z_4^{*2} + Z_5^{*2}}\right) = \Phi(-3.19785) = 0.000692289. \tag{33}$$

4.4.8 Probability of failure using the proposed method

In the proposed method (level 3) we calculate the probability of failure using the Gauss-Legendre recursion Expression (14), with the transformed failure region $g_U(u_1, u_2, \ldots, u_n)$, see (28)

$$-\mu_P - \sigma_P\Phi^{-1}(1-u_1) + \mu_s + \sigma_s\Phi^{-1}(u_2) - \mu_L - \sigma_L\Phi^{-1}(1-u_3) + b_0 + \sigma_b\Phi^{-1}(u_4) + 2c_0 + 2\sigma_c\Phi^{-1}(u_5) + \log\frac{2}{3}.$$

Table 3 shows the obtained results when we use the Gauss-Legendre recursion Expression (14) using the numerical values in Table 2 and the optimized method. Note that a polynomial of degree 50 is required for an almost exact approximation of the exact probability of failure.

Remark 1 *The coincidence of failure probabilities in this example is due to the particular selection of log-normal distributions, that was selected in order to be able to calculate the exact probability of failure, thus allowing a comparison between the exact and the calculated probabilities. If this is not the case, probabilities based on the classical method, FORM, SORM and the proposed method lead to different results (values of Pf).*

4.5 Case 2: Normal distributions

4.5.1 Assumptions for the random variables involved

In order to obtain different probabilities of failure associated with the FORM, SORM and the proposed method, we assume in this case that P, s, L, b and c are independent random normal distributions:

$$\begin{aligned} P &\sim N(\mu_P, \sigma_P^2), \\ s &\sim N(\mu_s, \sigma_s^2), \\ L &\sim N(\mu_{L_0}, \sigma_L^2), \\ b &\sim N(\mu_{b_0}, \sigma_b^2), \\ c &\sim N(\mu_{c_0}, \sigma_c^2). \end{aligned}$$

4.5.2 Level 2: FORM approximation

FORM methods require transforming the initial set of random variables to a set $\mathbf{Z}$ of standard independent $N(0,1)$ random variables

$$Z_1 = \frac{P-\mu_P}{\sigma_P}; \quad Z_2 = \frac{s-\mu_s}{\sigma_s}; \quad Z_3 = \frac{L-\mu_L}{\sigma_L}; \quad Z_4 = \frac{b-\mu_b}{\sigma_b}; \quad Z_5 = \frac{c-\mu_c}{\sigma_c}, \tag{34}$$

Thus, replacing (34) in the failure region (18), we get

$$g_Z(\mathbf{Z}) = \mu_s + \sigma_s Z_2 - \frac{3(\mu_P + \sigma_P Z_1)(\mu_L + \sigma_L Z_3)}{2(\mu_b + \sigma_b Z_4)(2\mu_c + 2\sigma_c Z_5)^2} \leq 0, \tag{35}$$

4.5.3 Level 3: Proposed method

In this subsubsection we determine the probability of failure using the proposed method and algorithm. Thus, we proceed as follows:

Step 1: We use the Rosenblatt transformation to transform the initial set of variables into a set of independent uniform $U(0,1)$ random variables V. To this end, we let:

$$\begin{array}{lll} P = & X_1 = & F^{-1}_{N(\mu_P,\sigma_P^2)}(V_1) = \mu_P + \sigma_P \Phi^{-1}(V_1) \\ s = & X_2 = & F^{-1}_{N(\mu_s,\sigma_s^2)}(V_2) = \mu_s + \sigma_s \Phi^{-1}(V_2) \\ L = & X_3 = & F^{-1}_{N(\mu_L,\sigma_L^2)}(V_3) = \mu_L + \sigma_L \Phi^{-1}(V_3) \\ b = & X_4 = & F^{-1}_{N(\mu_b,\sigma_b^2)}(V_4) = \mu_b + \sigma_b \Phi^{-1}(V_4) \\ c = & X_5 = & F^{-1}_{N(\mu_c,\sigma_c^2)}(V_5) = \mu_c + \sigma_c \Phi^{-1}(V_5) \end{array} \tag{36}$$

that is,

$$\begin{aligned} V_1 &= F_{(\mu_P,\sigma_P^2)}(X_1) = \Phi\left(\frac{X_1-\mu_P}{\sigma_P}\right) \\ V_2 &= F_{(\mu_s,\sigma_s^2)}(X_2) = \Phi\left(\frac{X_2-\mu_s}{\sigma_s}\right) \\ V_3 &= F_{(\mu_L,\sigma_L^2)}(X_3) = \Phi\left(\frac{X_3-\mu_L}{\sigma_L}\right) \\ V_4 &= F_{(\mu_b,\sigma_b^2)}(X_4) = \Phi\left(\frac{X_4-\mu_b}{\sigma_b}\right) \\ V_5 &= F_{(\mu_c,\sigma_c^2)}(X_5) = \Phi\left(\frac{X_5-\mu_c}{\sigma_c}\right) \end{aligned} \tag{37}$$

And the transformed failure region in terms of the V random variables becomes

$$\mu_s + \sigma_s \Phi^{-1}(V_2) - \frac{3 * (\mu_P + \sigma_P \Phi^{-1}(V_1)) * (\mu_L + \sigma_L \Phi^{-1}(V_3))}{2 * (\mu_b + \sigma_b \Phi^{-1}(V_4)) * (2\mu_c + 2\sigma_c \Phi^{-1}(V_5))^2} \leq 0. \tag{38}$$

Step 2: We transform the variables $\mathbf{V}$ such that the vector $(0,0,,\ldots,0)$ becomes the worst possible vector. To this end, we let $U_1 = 1 - V_1$ and $U_3 = 1 - V_3$. Thus, we get the failure region

$$\begin{array}{lll} U_1 &=& 1 - V_1 \\ U_2 &=& V_2 \\ U_3 &=& 1 - V_3 \\ U_4 &=& V_4 \\ U_5 &=& V_5 \end{array} \tag{39}$$

Step 3: Since the new variables U are iid uniform $U(0,1)$, we get

$$f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n) = 1;\ 0 \leq u_1 \leq 1;\ i = 1,2,3,4,5,$$

and the transformed failure region

$$\mu_s + \sigma_s \Phi^{-1}(U_2) - \frac{3(\mu_P + \sigma_P \Phi^{-1}(1-U_1))(\mu_L + \sigma_L \Phi^{-1}(1-U_3))}{2(\mu_b + \sigma_b \Phi^{-1}(U_4))(2\mu_c + 2\sigma_c \Phi^{-1}(U_5))^2} \leq 0. \tag{40}$$

Step 4: We calculate the probability of failure using the Gauss-Legendre recursion Expression (14).

4.5.4 Numerical example

In this numerical example we assume that P, s, L, b and c are independent random variables such that their means and standard deviations coincide with those in the Case 1, i.e., they can be obtained from those in case 1 using the expression:

$$\begin{array}{rl} \mu_i = & \exp(\mu_i^* + \sigma_i^{*2}/2) \\ \sigma_i^2 = & \exp(\mu_i^*)\left[\exp(2\sigma_i^{*2}) - \exp(\sigma_i^{*2})\right] \end{array} \tag{41}$$

where μ_i^* and σ_i^* are the means and the standard deviations for case 1. The subindex i takes the values P, s, L, b and c. The values are shown in Table 4,

4.5.5 FORM probability of failure

The FORM approximation requires transforming the initial set of random variables to a set $\mathbf{Z}$ of standard independent $N(0,1)$ random variables (see (34)).

The usual design selects the design point P^* such that its distance to the origin, in the transformed domain (the one associated with standard normal variables), is a minimum. In this example this point is

$$P^* = (Z_1^*, Z_2^*, Z_3^*, Z_4^*, Z_5^*) = (3.13736, -2.31931, 0.496413, -0.241195, -0.483687),$$

which leads to the design values(see (34))

$$\begin{array}{rcccl} P^* & = & \mu_P + Z_1^* \sigma_P & = & 753879N \\ s^* & = & \mu_s + Z_2^* \sigma_s & = & 20.126Mp \\ L^* & = & \mu_L + Z_3^* \sigma_L & = & 10.117m \\ b^* & = & \mu_b + Z_4^* \sigma_b & = & 0.5745m \\ c^* & = & \mu_c + Z_5^* \sigma_c & = & 0.9947m \end{array}$$

and to the following partial security coefficients

$$\begin{array}{rcccl} \gamma_p & = & \mu_P + P * \sigma_P/\mu_p & = & 1.8231 \\ \gamma_s & = & \mu_s/\mu_s + s * \sigma_s & = & 1.2467 \\ \gamma_L & = & \mu_L + L * \sigma_L/\mu_L & = & 1.0114 \\ \gamma_b & = & \mu_b/\mu_b + b * \sigma_b & = & 1.0028 \\ \gamma_c & = & \mu_c/\mu_c + c * \sigma_c & = & 1.0054 \end{array} \tag{42}$$

and to a global value of $\gamma_G = \gamma_P \gamma_s \gamma_L \gamma_b \gamma_c = 2.317$.

Table 4: Means and standard deviations of the 5 random variables involved in the practical case 2.

Variable	Mean	Standard deviation
P	$\mu_P = 413527$	$\sigma_P = 108483$
s	$\mu_s = 25091184.69$	$\sigma_s = 2140947.96$
L	$\mu_L = 10.0027$	$\sigma_L = 0.23$
b	$\mu_b = 0.576035$	$\sigma_b = 0.00635$
c	$\mu_c = 1$	$\sigma_c = 0.011033$

Table 5: Different estimated values of the probability of failure for different approximations of the Gauss Legendre quadrature using the optimized method for example 2.

n	m	Estimated probability	cpu time (in ticks)
5	10	0.000000	14
5	20	0.000009	252
5	40	0.000024	5970
5	60	0.000029	47234
5	80	0.000031	219468
5	100	0.000032	753609

Note that the partial safety coefficients, γ_L, γ_b and γ_c, for L, b and c, respectively, are small, as expected because their assumed standard deviations are small.

The associated probability of failure is:

$$P_f = \Phi(-\beta) = \Phi\left(-\sqrt{Z_1^{*2} + Z_2^{*2} + Z_3^{*2} + Z_4^{*2} + Z_5^{*2}}\right) = \Phi(-3.96999) = 0.0000359377. \tag{43}$$

4.5.6 Probability of failure using the proposed method

In the proposed method (level 3) we calculate the probability of failure using the Gauss-Legendre recursion Expression (14), with the transformed failure region $g_U(u_1, u_2, \ldots, u_n)$ is (see (40))

$$\mu_s + \sigma_s \Phi^{-1}(u_2) - \frac{3(\mu_P + \sigma_P \Phi^{-1}(1-u_1))(\mu_L + \sigma_L \Phi^{-1}(1-u_3))}{2(b_0 + \sigma_b \Phi^{-1}(u_4))(c_0 + 2\sigma_c \Phi^{-1}(u_5))^2}.$$

Table 5 shows the obtained results when we use the Gauss-Legendre recursion Expression (14) using the numerical values in Table 4 and the optimized method.

5 Conclusions

The proposed method based on transforming the initial set of variables to one with uniform marginals combined with the Gauss-Legendre quadrature formula allows calculating the multiple integral associated with the probability of failure of some reliability problems. The method is especially useful for approximating tail probabilities, where other methods fail. The degree of the polynomial can be selected by increasing its value until convergence is achieved.

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