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E. Castillo, A. J. Conejo, R. Mínguez and C. Castillo

Corresponding Author:

Prof. Antonio J. Conejo
Department of Electrical Engineering
ETSI Industriales
Univ. Castilla - La Mancha
Campus Universitario s/n
13071 Ciudad Real
Spain

Phone: + 34 926 295433
Fax: + 34 926 295361

Email: Antonio.Conejo@uclm.es

A Closed Formula for Local Sensitivity Analysis in Mathematical Programming

Enrique Castillo[1], **Antonio J. Conejo**[2],
Roberto Mínguez[3] **and Carmen Castillo**[4]

[1]*Department of Applied Mathematics and Computational Sciences,*
Univ. of Cantabria, Santander, Spain
[2]*Department of Electrical Engineering, Univ. of Castilla-La Mancha, Ciudad Real, Spain*
[3]*Department of Civil Engineering, Univ. of Castilla-La Mancha, Ciudad Real, Spain*
[4] *Grupo de Puertos y Costas, CEAMA, Univ. of Granada, Spain*

Abstract

This paper introduces a method for local sensitivity analysis of practical interest. A theorem and tools are given that provide analytical formulas for the local sensitivities of the objective function optimal value with respect to parameters. The method is based on the well known duality property of mathematical programming that states that the partial derivatives of the primal objective function with respect to the constraints right hand side parameters are the optimal values of the dual problem variables. For the parameters or data, for which sensitivities are sought, to appear on the right hand side, they are converted into artificial variables and locked to their actual values, thus obtaining the desired constraints. If the problem is degenerated and partial derivatives do not exist, the method also permits obtaining the right and left derivatives if they exists. In addition to its general applicability, the method is also computationally inexpensive because the necessary information becomes available without extra calculations. Moreover, analytical relations among sensitivities, locally valid, are straightforwardly obtained. If sensitivities are needed in relation to any constraint and not to the objective function, an equivalent problem is formulated that allows obtaining such sensitivities. The method is illustrated by its application to two examples, one degenerated and one of a competitive market.

Key Words: Local sensitivity, Mathematical Programming, Duality.

1 Introduction and Motivation

In engineering practice, one has to work frequently with mathematical models to describe reality or make decisions. However, mathematical models are simplifications of reality. Nevertheless, when we collect data and specify a model, we often act as if the model is true and the associated assumptions are valid. More often than not, conclusions drawn from an analysis are sensitive to changes in a model, deviations from assumptions, or other perturbations in the inputs. Thus, one needs to know the influence of each data item on the final results so as to make the adequate corrections when necessary. It is therefore essential for data analysts to be able to assess the sensitivity of their conclusions to various perturbations in the inputs. This is known as sensitivity analysis. The motivation for sensitivity analysis is clear. It allows the analyst to assess the effects on inferences of departures from the assumptions made and the data values, detect outliers or wrong data values, define testing strategies, optimize resources, reduce costs, and avoid unexpected problems.

There is a large literature on sensitivity analysis and outlier detection; see, for example, the books by Hawkins (1980), Belsley, Kuh, and Welsch (1980), Cook and Weisberg, (1982), Chatterjee and Hadi (1988), and Barnett and Lewis (1994), and the papers by Gray (1986), Cook(1986), Schwarzmann (1991), Paul and Fung (1991), Nyquist (1992), Hadi and Simonoff (1993), Billor, Chatterjee, and Hadi (2001), and Winsnowski, Montgomery, and James (2001).

In relation to optimization theory and applications, relevant references related to sensitivity analysis include Dempe (1993), Dinkel and Tretter (1993), Ralph and Dempe (1995), Wagner (1995), Levy (1996), Marcotte and Zhu (1996), Wallace (2000), Dawande and Hooker (2000), and Levy and Mordukhovich (2004).

As an example of engineering applications, relevant references on power engineering related to sensitivity analysis include Almeida and Salgado (2000), He et al. (2002), Proca and Keyhani (2002), Araujo Ferreira et al. (2002), Orfanogianni and Bacher (2003), Zarate and Castro (2004) and Chung et al. (2004).

In this paper we propose a method for sensitivity analysis that is based on the well known duality property of mathematical programming that states that the partial derivatives of the primal objective function with respect to the constraints right hand side parameters are the optimal values of the dual problem variables. The method is applicable to linear and nonlinear models. In addition to its general applicability, the method is also computationally inexpensive because the necessary information becomes available without extra calculations. Moreover, analytical relations among sensitivities, locally valid, are straightforwardly obtained. If sensitivities are needed in relation to any constraint and not to the objective function, an equivalent problem is formulated that allows obtaining such sensitivities.

The main contributions of this paper are:

1. Under local convexity assumptions, it provides a general procedure to compute the sensitivities of the objective function with respecto to any parameter of a general nonlinear programming problem, when they exist.

2. When the local sensitivities, partial derivatives, do not exist, the proposed method allows obtaining left and right derivatives if they exist.

3. It also provides a linear system of equations that relates the actual sensitivities in the neighborhood of the optimal solution.

4. Once the optimal solution of the general nonlinear programming problem is found, the paper provides an efficient yet simple way to compute the sensitivities of any constraint with respect to any parameter.

The paper is structured as follows. In Section 2 primal and dual problems are described and then it is explained how some dual variables provide the sensitivities of the objective function to changes in the constraints. In Section 3 it is shown how the solution of the dual problem can be obtained from the solution of the primal problem, and the linear relationship among sensitivities in the neighborhood or the optimal solution is derived. The proposed method for obtaining the sensitivity of the optimal value of the objective function with respect to small changes in the data is introduced in Section 4 in its general form. To this end, the optimization method is modified, using artificial variables and constraints, for the parameters whose sensitivities are sought, to appear on the right hand side of some constraints.

This leads to the main theorem of the paper that gives a closed expression for the sensitivities of the objective function optimal value with respect to any parameter or data value. The method is also valid for degenerated cases and is illustrated with an example. Section 5 describes how to obtain sensitivities related to any constraint and not the objective function, that is, the partial derivatives of any constraint with respect to parameters. The general applicability of the proposed method for sensitivity analysis is illustrated through a case study in Section 6. Finally, Section 7 offers some concluding remarks.

2 Background

In this section and for the sake of reminding the reader and introducing the notation we reproduce some well known material.

Consider the following general nonlinear *primal problem* (P):

$$\underset{\mathbf{x}}{\text{Minimize}} \ Z_P = f(\mathbf{x};\mathbf{a}) \tag{1}$$

subject to

$$\mathbf{h}(\mathbf{x};\mathbf{a}) = \mathbf{b}, \tag{2}$$

$$\mathbf{g}(\mathbf{x};\mathbf{a}) \leq \mathbf{c}, \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{a} \in \mathbb{R}^t$, $\mathbf{b} \in \mathbb{R}^p$, $\mathbf{c} \in \mathbb{R}^q$.

The Primal problem P, as stated in (1)–(3), has an associated dual problem D, which is defined as:

$$\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{Maximize}} \ Q_D = \ \text{Inf}_{\mathbf{x}} \ \{\mathcal{L}(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu};\mathbf{a},\mathbf{b},\mathbf{c})\} \tag{4}$$

subject to

$$\boldsymbol{\mu} \geq \mathbf{0}, \tag{5}$$

where

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu};\mathbf{a},\mathbf{b},\mathbf{c}) = f(\mathbf{x};\mathbf{a}) + \boldsymbol{\lambda}^T(\mathbf{h}(\mathbf{x};\mathbf{a}) - \mathbf{b}) + \boldsymbol{\mu}^T(\mathbf{g}(\mathbf{x};\mathbf{a}) - \mathbf{c}), \tag{6}$$

is the Lagrangian function associated with the primal problem (1)–(3), and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the dual variables, are vectors of dimensions p and q, respectively. Note that only the dual variables ($\boldsymbol{\mu}$ in this case) associated with the inequality constraints ($\mathbf{g}(\mathbf{x})$ in this case), must be nonnegative.

Given some regularity conditions on local convexity (see Luenberger (1989)), if the primal problem (1)–(3) has a locally optimal solution $\mathbf{x}^*$, the dual problem (4)–(5) also has a locally optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, and the optimal values of the objective functions of both problems coincide.

The primal and the dual problems (1)–(3) and (4)–(5), respectively, can be solved using the Karush-Kuhn-Tucker first order necessary conditions (KKTCs) (see, for example, Luenberger (1989), Bazaraa, Sherali and Shetty (1993), Castillo et al. (2001)):

$$\begin{aligned} \nabla_{\mathbf{x}} f(\mathbf{x}^*; \mathbf{a}) + \boldsymbol{\lambda}^{*T} \nabla_{\mathbf{x}} \mathbf{h}(\mathbf{x}^*; \mathbf{a}) + \boldsymbol{\mu}^{*T} \nabla_{\mathbf{x}} \mathbf{g}(\mathbf{x}^*; \mathbf{a}) &= \mathbf{0}, & (7) \\ \mathbf{h}(\mathbf{x}^*; \mathbf{a}) - \mathbf{b} &= \mathbf{0}, & (8) \\ \mathbf{g}(\mathbf{x}^*; \mathbf{a}) - \mathbf{c} &\leq \mathbf{0}, & (9) \\ \boldsymbol{\mu}^{*T} (\mathbf{g}(\mathbf{x}^*; \mathbf{a}) - \mathbf{c}) &= \mathbf{0}, & (10) \\ \boldsymbol{\mu}^* &\geq \mathbf{0}, & (11) \end{aligned}$$

where $\mathbf{x}^*$ and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ are the primal and dual optimal solutions, $\nabla_{\mathbf{x}} f(\mathbf{x}^*; \mathbf{a})$ is the vector of partial derivatives of $f(\mathbf{x}^*; \mathbf{a})$ with respect to $\mathbf{x}$, evaluated at the optimal value $\mathbf{x}^*$. The vectors $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*$ are also called the *Kuhn–Tucker multipliers.* Condition (7) says that the gradient of the Lagrangian function in (6) evaluated at the optimal solution $\mathbf{x}^*$ must be zero. Conditions (8) and (9) are *the primal feasibility* conditions. Condition (10) is the *complementary slackness condition.* Condition (11) requires the nonnegativity of the multipliers of the inequality constraints and is referred to as the *dual feasibility conditions.*

The practical importance of the dual solutions derives from the fact that the values of the dual variables give the sensitivities of the optimal objective function value with respect to the parameters $\mathbf{b}$ and $\mathbf{c}$ appearing on the right hand side of the constraints. This is stated in Theorem 1 below (Luenberger, 1989).

Theorem 1 (Sensitivities) *Consider the optimization problem (1)–(3). Then, we* *have:*

$$\frac{\partial f(\mathbf{x}^*;\mathbf{a})}{\partial b_i} = -\lambda_i^*;\ \ i=1,2,\ldots,p;\ \ \frac{\partial f(\mathbf{x}^*;\mathbf{a})}{\partial c_j} = -\mu_j^*;\ \ j=1,2,\ldots,q,$$

i.e., the sensitivities of the optimal objective function value of the problem (1)–(3) with respect to changes in the terms appearing on the right hand side of the constraints are the negative of the optimal values of the corresponding dual variables.

The proof of this theorem can be found, for instance, in (Luenberger, 1989).

For this important result to be applicable to practical cases of sensitivity analysis, the parameters for which the sensitivities are sought must appear on the right hand side of the primal problem constraints. At this point the reader can ask him/herself and what about parameters not satisfying this condition? The answer to this question will be given by Theorem 2 in Section 4.

3 Obtaining the Set of All Dual Variable Values

Dual solutions are easily obtainable when solving the primal problem. In fact, when asked for the optimal solution of a primal problem, most algorithms embedded in computer packages (GAMS-SNOPT, GAMS-CONOPT, GAMS-MINOS, MATLAB, etc.) also give the optimal solution of the associated dual problem, with practically no extra computational cost. However, if one is interested in deriving analytical expressions for the optimal values of the dual variables or calculating these values, one can proceed as follows.

The KKTCs in (7)–(11) allow one to obtain the multipliers (values of the dual variables $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$) once the optimal solution $\mathbf{x}^*$ of the primal problem (1)–(3) has been obtained using the subset of linear equations in $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$:

$$\begin{aligned} \nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{a}) + \boldsymbol{\lambda}^{*T}\nabla_{\mathbf{x}}\mathbf{h}(\mathbf{x}^*;\mathbf{a}) + \boldsymbol{\mu}^{*T}\nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x}^*;\mathbf{a}) &= \mathbf{0}, && (12)\\ \boldsymbol{\mu}^{*T}(\mathbf{g}(\mathbf{x}^*;\mathbf{a}) - \mathbf{c}) &= \mathbf{0}, && (13)\\ \boldsymbol{\mu}^* &\geq \mathbf{0}, && (14) \end{aligned}$$

Then, we proceed with the following steps:

Step 1. Determine the subset of inequality constraints (3) which are active, i.e., those j such that $g_j(\mathbf{x}^*; \mathbf{a}) = c_j$. Let J be this set, and let $\mu_j = 0$ for all $j \notin J$.

Step 2. Solve in $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ the system of equations and inequalities:

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*; \mathbf{a}) + \boldsymbol{\lambda}^{*T} \nabla_{\mathbf{x}} \mathbf{h}(\mathbf{x}^*; \mathbf{a}) + \sum_{j \in J} \mu_j^* \nabla_{\mathbf{x}} g_j(\mathbf{x}^*; \mathbf{a}) = \mathbf{0}, \quad (15)$$

$$\mu_j^* \geq 0, \quad j \in J. \quad (16)$$

Since the unknowns in system (15) and (16) are the sensitivity vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, this system is linear, and therefore easy to solve. This system can be solved using the procedure stated in Castillo et al. (1999, 2004).

Furthermore, the solution of system (15) and (16) provides analytical expressions to calculate vectors $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ as a function of the primal solution $\mathbf{x}^*$ of problem (1)–(3). This is an important result because this analytical expression is not provided by practical solution algorithms.

Therefore, the solution of the dual problem is readily available after solving the primal problem, and since the solution to the dual problem are the sensitivities with respect to some parameters $\mathbf{b}$ and $\mathbf{c}$, these sensitivities are immediately available after the solution of the primal problem. However, nothing is known about the sensitivities with respect to the parameter $\mathbf{a}$. The following section uses a simple trick to obtain these sensitivities.

4 A General Formula for Local Sensitivity Analysis

Two ideas are utilized here. The first is that local sensitivity can be measured by the partial derivatives of the objective function value with respect to the parameter or the data whose sensitivity is sought. The second, as we have seen in Section 2, is that the solutions to the primal and dual problems are related, and the solutions of the dual problem provide the sensitivities of the objective function values of the primal with respect to changes in the right hand side values of its constraints. These two ideas are combined, using the following theorem that provides a general method for sensitivity analysis.

Theorem 2 (Objective function sensitivities with respect to the parameter a) *The sensitivity of the objective function of the primal problem (1)–(3) with respect to the parameter* $\mathbf{a}$ *is given by*

$$\nabla_{\mathbf{a}}\mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*; \mathbf{a}, \mathbf{b}, \mathbf{c}), \tag{17}$$

which is the partial derivative of its Lagrangian function

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{x}; \mathbf{a}) + \boldsymbol{\lambda}^T(\mathbf{h}(\mathbf{x}; \mathbf{a}) - \mathbf{b}) + \boldsymbol{\mu}^T(\mathbf{g}(\mathbf{x}; \mathbf{a}) - \mathbf{c}), \tag{18}$$

with respect to $\mathbf{a}$ *evaluated at the optimal solution* $\mathbf{x}^*, \boldsymbol{\lambda}^*$*, and* $\boldsymbol{\mu}^*$*.*

Proof. The problem (1)–(3) can be written in the equivalent form:

$$\underset{\mathbf{x}, \tilde{\mathbf{a}}}{\text{Minimize}} \quad Z_P = f(\mathbf{x}; \tilde{\mathbf{a}}) \tag{19}$$

subject to

$$\begin{aligned} \mathbf{h}(\mathbf{x}; \tilde{\mathbf{a}}) &= \mathbf{b}; \;\; \boldsymbol{\lambda} && (20) \\ \mathbf{g}(\mathbf{x}; \tilde{\mathbf{a}}) &\leq \mathbf{c}; \;\; \boldsymbol{\mu} && (21) \\ \tilde{\mathbf{a}} &= \mathbf{a}; \;\; \boldsymbol{\eta} && (22) \end{aligned}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are the corresponding dual variables (multipliers).

Since the data $\mathbf{a}$ appears on the right hand side of (22), the values of the corresponding dual variables ($\boldsymbol{\eta}$) give the sensitivities of the objective function with respect to $\mathbf{a}$. The KKTC for the problem (19)–(22) are:

$$\begin{aligned} \nabla_{\mathbf{x},\tilde{\mathbf{a}}} f(\mathbf{x}^*; \tilde{\mathbf{a}}^*) + \boldsymbol{\lambda}^{*T} \nabla_{\mathbf{x},\tilde{\mathbf{a}}} \mathbf{h}(\mathbf{x}^*; \tilde{\mathbf{a}}^*) + & \\ \boldsymbol{\mu}^{*T} \nabla_{\mathbf{x},\tilde{\mathbf{a}}} \mathbf{g}(\mathbf{x}^*; \tilde{\mathbf{a}}^*) + \boldsymbol{\eta}^{*T} \nabla_{\mathbf{x},\tilde{\mathbf{a}}^*} \tilde{\mathbf{a}}^* &= \mathbf{0}, && (23) \end{aligned}$$

$$
\begin{aligned}
\mathbf{h}(\mathbf{x}^*;\tilde{\mathbf{a}}^*) - \mathbf{b} &= \mathbf{0}, && (24)\\
\mathbf{g}(\mathbf{x}^*;\tilde{\mathbf{a}}^*) - \mathbf{c} &\leq \mathbf{0}, && (25)\\
\boldsymbol{\mu}^{*T}(\mathbf{g}(\mathbf{x}^*;\tilde{\mathbf{a}}^*) - \mathbf{c}) &= \mathbf{0}, && (26)\\
\tilde{\mathbf{a}} &= \mathbf{a}, && (27)\\
\boldsymbol{\mu}^* &\geq \mathbf{0}. && (28)
\end{aligned}
$$

From (23) we get

$$
\begin{aligned}
\nabla_{\tilde{\mathbf{a}}}\mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*;\tilde{\mathbf{a}},\mathbf{b},\mathbf{c}) &= \nabla_{\tilde{\mathbf{a}}}f(\mathbf{x}^*;\tilde{\mathbf{a}}) + \boldsymbol{\lambda}^{*T}\nabla_{\tilde{\mathbf{a}}}\mathbf{h}(\mathbf{x}^*;\tilde{\mathbf{a}}) + \boldsymbol{\mu}^{*T}\nabla_{\tilde{\mathbf{a}}}\mathbf{g}(\mathbf{x}^*;\tilde{\mathbf{a}})\\
&= -\boldsymbol{\eta}^*, \qquad (29)
\end{aligned}
$$

i.e., the sensitivity of the objective function of the problem (1)–(3) with respect to the parameter $\mathbf{a}$ is the partial derivative of its Lagrangian function with respect to $\mathbf{a}$ at the optimal point. ∎

It is clear that problems (1)–(3) and (19)–(22) are equivalent, but for the second the sensitivities with respect to $\mathbf{a}$ are readily available. Note that to be able to use the important result of Theorem 1, we convert the data $\mathbf{a}$ into artificial variables $\tilde{\mathbf{a}}$ and set them to their actual values $\mathbf{a}$ as in constraint (27). Then, by Theorem 2, the values of the dual variables associated with (27) are the sensitivities sought after, i.e., the partial derivatives $\partial Z_P/\partial a_i; i = 1, 2, \ldots, t$.

4.1 Linear Programming Case

The simplest case where Theorem 2 can be applied is the case of linear programming.

Consider the following linear programming problem:

$$
\underset{x_1, x_2, \ldots, x_m}{\text{Minimize}} \quad \sum_{i=1}^{m} c_i x_i \qquad (30)
$$

subject to

$$
\begin{aligned}
\sum_{i=1}^{m} a_{ji}x_i &= r_j; \quad j = 1, 2, \ldots, p && (31)\\
\sum_{i=1}^{m} b_{ki}x_i &\leq s_k; \quad k = 1, 2, \ldots, q && (32)
\end{aligned}
$$

The Lagrangian function becomes

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \mathbf{c}, \mathbf{r}, \mathbf{s}) = \sum_{i=1}^{m} c_i x_i + \sum_{j=1}^{p} \lambda_j \left(\sum_{i=1}^{m} a_{ji} x_i - r_j \right) + \sum_{k=1}^{q} \mu_k \left(\sum_{i=1}^{m} b_{ki} x_i - s_k \right) \quad (33)$$

To obtain the sensitivities of the optimal value of the objective function to r_t, s_t, c_t, $a_{t\ell}$ or $b_{t\ell}$, following Theorem 2, we simply obtain the partial derivatives of the Lagrangian function with respect to the corresponding parameter, that is,

$$\frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial r_t} = -\lambda_t^* \quad (34)$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial s_t} = -\mu_t^* \quad (35)$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial c_t} = x_t^* \quad (36)$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial a_{t\ell}} = \lambda_t^* x_\ell^* \quad (37)$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial b_{t\ell}} = \mu_t^* x_\ell^* \quad (38)$$

This is a simple case that leads to very neat results.

Consider also the case of all parameters depending on a common parameter u, i.e., the problem

$$\underset{x_1, x_2, \ldots, x_m}{\text{Minimize}} \quad \sum_{i=1}^{m} c_i(u) x_i \quad (39)$$

subject to

$$\sum_{i=1}^{m} a_{ji}(u) x_i = r_j(u); \quad j = 1, 2, \ldots, p \quad (40)$$

$$\sum_{i=1}^{m} b_{ki}(u) x_i \leq s_k(u); \quad k = 1, 2, \ldots, q \quad (41)$$

Then, the sensitivity of the optimal value of the objective function to u is given by (see Equation (33)):

$$\begin{aligned} \frac{\partial \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}; u)}{\partial u} &= \sum_{i=1}^{m} \frac{d\, c_i(u)}{du} x_i + \sum_{j=1}^{p} \lambda_j \left(\sum_{i=1}^{m} \frac{d\, a_{ji}(u)}{du} x_i - \frac{d\, r_j(u)}{du} \right) \\ &\quad + \sum_{k=1}^{q} \mu_k \left(\sum_{i=1}^{m} \frac{d\, b_{ki}(u)}{du} x_i - \frac{d\, s_k(u)}{du} \right) \end{aligned} \quad (42)$$

Note that the cases in (34) to (38) are particular cases of (42).

Note that Theorem 2 is applicable to any nonlinear programming problem as straightforwardly as to the linear programming problem above.

4.2 Illustrative Example

In this section we include a degenerate linear problem with the aim of illustrating the power of the proposed method for evaluating the sensitivities, even if partial derivatives are not defined due to degeneracy.

Consider the degenerate problem

$$\underset{x_1, x_2}{\text{Minimize}} \quad Z_P = x_2$$

subject to

$$\begin{array}{rrcr} ax_1 & -x_2 & \leq & 0 \\ -x_1 & & \leq & -a \\ -x_1 & -x_2 & \leq & -b \end{array} \tag{43}$$

whose optimal solution for $a = 1$ and $b = 2$ is

$$x_1^* = x_2^* = 1; \quad Z_P^* = 1,$$

which is illustrated in the upper part of Figure 1.

FIGURE 1 GOES ABOUT HERE

Its Lagrangian function is

$$\mathcal{L}(x_1, x_2; \mu_1, \mu_2, \mu_3) = x_2 + \mu_1(ax_1 - x_2) + \mu_2(a - x_1) + \mu_3(b - x_1 - x_2).$$

From the KKT conditions (12)-(14) we get

$$\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} \mu_1^* & \mu_2^* & \mu_3^* \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}; \quad \mu_1^*, \mu_2^*, \mu_3^* \geq 0$$

which solution is (see Padberg (1995), Castillo et al. (1999) and Castillo and Jubete (2004))

$$\begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} (1+\lambda)/2 \\ \lambda \\ (1-\lambda)/2 \end{pmatrix}; \quad 0 \le \lambda \le 1.$$

and applying Theorem 2 we obtain the sensitivities

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*; \mu_1^*, \mu_2^*, \mu_3^*)}{\partial a} = \mu_1^* x_1^* + \mu_2^* = \mu_1^* + \mu_2^* = (1+3\lambda)/2 \quad (44)$$

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*; \mu_1^*, \mu_2^*, \mu_3^*)}{\partial b} = \mu_3^* = (1-\lambda)/2 \quad (45)$$

Note that the sensitivity of the objective function with respect to a is not constant but ranges from 1/2 to 2. This implies that the corresponding partial derivative does not exist. However, the right and left derivatives exist and take those extreme values. Note that $(a = 1, b = 2)$

$$\begin{aligned} \frac{\partial Z_P}{\partial a_+} &= \frac{\partial a^2}{\partial a} = 2a = 2 \\ \frac{\partial Z_P}{\partial a_-} &= \frac{\partial \left(\frac{ab}{1+a}\right)}{\partial a} = \frac{b}{(1+a)^2} = 1/2 \end{aligned}$$

This is illustrated in Figure 1, where the difference in the right, 2, and left, 1/2, partial derivatives is due to the change of the active constraints.

Similarly, the sensitivity of the objective function with respect to b is not constant but ranges from 0 to 1/2. Note that $(a = 1, b = 2)$

$$\begin{aligned} \frac{\partial Z_P}{\partial b_+} &= \frac{\partial \left(\frac{ab}{1+a}\right)}{\partial b} = \frac{a}{1+a} = 1/2 \\ \frac{\partial Z_P}{\partial b_-} &= \frac{\partial a}{\partial b} = 0 \end{aligned}$$

This is illustrated in Figure 2, where the difference in the right, 1/2, and left, 0, partial derivatives is again due to the change of the active constraints.

FIGURE 2 GOES ABOUT HERE

This example illustrates how the proposed method allows determining not only the partial derivatives of the optimal objective function value to parameters but determining if this derivative does not exist and the values of the right and left partial derivatives.

5 Sensitivities Related to Any Active Constraint

Consider the following convenient reformulation of problem problem (1)–(3):

$$\underset{\mathbf{x}}{\text{Minimize}} \quad Z_P = f(\mathbf{x}) \tag{46}$$

subject to

$$\begin{aligned} h_k(\mathbf{x}) &= 0; \quad k = 1, 2, \dots, p & (47)\\ g_j(\mathbf{x}) &\le 0; \quad j = 1, 2, \dots, q & (48) \end{aligned}$$

where $f : \mathbb{R}^m \to \mathbb{R}$, $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^p$, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^q$ with $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \dots, h_p(\mathbf{x}))^T$ and $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_q(\mathbf{x}))^T$ are regular functions over the feasible region $S = \{\mathbf{x} | \mathbf{h}(\mathbf{x}) = \mathbf{0}, \mathbf{g}(\mathbf{x}) \le \mathbf{0}\}$.

The Karush-Kuhn-Tucker (KKTCs) first order optimality conditions for the NLPP (46)-(48) are:

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{k=1}^{p} \lambda_k^* \nabla h_k(\mathbf{x}^*) + \sum_{j=1}^{q} \mu_j^* \nabla g_j(\mathbf{x}^*) &= \mathbf{0} & (49)\\ h_k(\mathbf{x}^*) &= 0, k = 1, \dots, p & (50)\\ g_j(\mathbf{x}^*) &\le 0, j = 1, \dots, q & (51)\\ \mu_j^* g_j(\mathbf{x}^*) &= 0, j = 1, \dots, q & (52)\\ \mu_j^* &\ge 0, j = 1, \dots, q & (53) \end{aligned}$$

Assume also that no redundant constraints exist and that the NLPP has an optimal solution $\mathbf{x}^*$ such that $Z_P^* = f(\mathbf{x}^*)$. Then conditions (49)-(53) hold. If $\lambda_1 > 0$, then we can write

$$\nabla h_1(\mathbf{x}^*) + \frac{1}{\lambda_1^*} \nabla f(\mathbf{x}^*) + \sum_{k=2}^{\ell} \frac{\lambda_k^*}{\lambda_1^*} \nabla h_k(\mathbf{x}^*) + \sum_{j=1}^{m} \frac{\mu_j^*}{\lambda_1^*} \nabla g_j(\mathbf{x}^*) = \mathbf{0} \tag{54}$$

$$
\begin{aligned}
f(\mathbf{x}^*) - Z_P^* &= 0 && (55)\\
h_k(\mathbf{x}^*) &= 0, k = 2, \ldots, p && (56)\\
g_j(\mathbf{x}^*) &\leq 0, j = 1, \ldots, q && (57)\\
\frac{\mu_j^*}{\lambda_1^*} g_j(\mathbf{x}^*) &= 0, j = 1, \ldots, q && (58)\\
\frac{\mu_j^*}{\lambda_1^*} &\geq 0, j = 1, \ldots, q && (59)
\end{aligned}
$$

that are the KKTC for the problem

$$
\underset{\mathbf{x}}{\text{Minimize}} \quad Z_P = h_1(\mathbf{x}) \tag{60}
$$

subject to

$$
\begin{aligned}
f(\mathbf{x}) - Z_P^* &= 0 && (61)\\
h_k(\mathbf{x}) &= 0; \quad k = 2, 3, \ldots, p && (62)\\
g_j(\mathbf{x}) &\leq 0; \quad j = 1, 2, \ldots, q && (63)
\end{aligned}
$$

The multipliers in (54) give the corresponding sensitivities of $h_1(\mathbf{x})$ to changes in Z_P^*, h_k $(k = 2, 3, \ldots, p)$ and g_j $(k = 1, 2, \ldots, q)$, respectively.

For $\lambda_1^* < 0$, we obtain similar results but a maximization problem. The μ^* multipliers can be dealt with in the same form as the case $\lambda_1^* > 0$, i.e., if the associated constraints are active.

The results in this section, apart from giving the sensitivities of a constraint, allow solving the initial problem in an alternative way. This has advantages in some cases in which one looks for optimal solutions of the initial problem (46)-(48) with a given optimal value and wants to fix the right hand side of a constraint for that to be possible. In this case, the alternative statement (60)-(63) allows solving the problem in one step, while the initial problem requires iterations.

6 Case Study

A producer sells its production p_t $(t = 1, \ldots, T)$ in a multi-period spot market whose prices λ_t $(t = 1, \ldots, T)$ are random variables. These random variables are characterized by their forecast average values, λ_t^{avg}, $(t = 1, \ldots, T)$ and an estimate of their $T \times T$ covariance matrix Q^{est}.

If the production cost of producing the quantity p_t at time t is given by $b\, p_t + \frac{1}{2} a\, p_t^2$, the expected profit P for the producer from selling its production in the spot market is

$$P = E_{\lambda_1, \ldots, \lambda_T} \left\{ \sum_{t=1}^{T} \left[\lambda_t\, p_t - b\, p_t - \frac{1}{2} a\, p_t^2 \right] \right\} = \sum_{t=1}^{T} \left[\lambda_t^{\text{avg}}\, p_t - b\, p_t - \frac{1}{2} a\, p_t^2 \right] \tag{64}$$

where a and b are positive cost coefficients.

On the other hand, the variance of the profit V (a risk measure) is given by

$$V = \text{Var}_{\lambda_1, \ldots, \lambda_T} \left\{ \sum_{t=1}^{T} \left[\lambda_t\, p_t - b\, p_t - \frac{1}{2} a\, p_t^2 \right] \right\} = \text{Var}_{\lambda_1, \ldots, \lambda_T} \left\{ \sum_{t=1}^{T} \lambda_t\, p_t \right\} = \sum_{i=1}^{T} \sum_{j=1}^{T} Q_{ij}^{\text{est}} p_i\, p_j \tag{65}$$

The producer operating constraints include maximum capacity, and limits on increasing/decreasing production during two consecutive time periods, i.e.,

$$p^{\min} \leq p_t \leq p^{\max}; \quad t = 1, \ldots, T \tag{66}$$

where $p^{\min}$ and $p^{\max}$ are the minimum and maximum productions levels, respectively, and

$$p_t - p_{t-1} \leq r^{\text{up}}; \quad t = 1, \ldots, T \tag{67}$$

$$p_{t-1} - p_t \leq r^{\text{down}}; \quad t = 1, \ldots, T \tag{68}$$

where p_0 is the production during the time period previous to the study horizon, and r^{up} and r^{down} are the bounds on the increasing/decreasing production change, respectively.

The producer faces two problems: to maximize expected profits limiting the risk level, or to minimize risk ensuring a level of profit. These problems are formulated below.

The risk-constrained maximum profit problem is

$$\underset{p_t}{\text{Maximize}} \quad P = \sum_{t=1}^{T} \left[\lambda_t^{\text{avg}} \, p_t - b \, p_t - \tfrac{1}{2} a \, p_t^2 \right] \tag{69}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{T} \sum_{j=1}^{T} Q_{ij}^{\text{est}} p_i \, p_j &\leq R && (70)\\
p^{\min} \leq p_t &\leq p^{\max}; \quad t = 1, \ldots, T && (71)\\
p_t - p_{t-1} &\leq r^{\text{up}}; \quad t = 1, \ldots, T && (72)\\
p_{t-1} - p_t &\leq r^{\text{down}}; \quad t = 1, \ldots, T && (73)
\end{aligned}$$

The profit-constrained minimum risk problem is

$$\underset{p_t}{\text{Minimize}} \quad V = \sum_{i=1}^{T} \sum_{j=1}^{T} Q_{ij}^{\text{est}} p_i \, p_j \tag{74}$$

subject to

$$\begin{aligned}
\sum_{t=1}^{T} \left(\lambda_t^{\text{avg}} p_t - b \, p_t - \frac{1}{2} a \, p_t^2 \right) &\geq B = P^* && (75)\\
p^{\min} \leq p_t &\leq p^{\max}; \quad t = 1, \ldots, T && (76)\\
p_t - p_{t-1} &\leq r^{\text{up}}; \quad t = 1, \ldots, T && (77)\\
p_{t-1} - p_t &\leq r^{\text{down}}; \quad t = 1, \ldots, T && (78)
\end{aligned}$$

where P^* is the optimal value of the problem (69)-(73).

6.1 Sensitivity Analysis

In this section we apply Theorem 2 to the case study. Since we have two optimization problems we need to apply it twice.

The Lagrange function associated with the problem (69)-(73) is

$$L_1(\mathbf{p}, \mu^1, \boldsymbol{\mu}^2, \boldsymbol{\mu}^3, \boldsymbol{\mu}^4, \boldsymbol{\mu}^5) \quad = \quad \sum_{t=1}^{T} \left[\lambda_t^{\text{avg}} p_t - b p_t - \frac{1}{2} a p_t^2 \right] + \mu^1 \left(\sum_{i=1}^{T} \sum_{j=1}^{T} Q_{ij}^{\text{est}} p_i p_j - R \right)$$

$$\sum_{t=1}^{T}\mu_t^2\left[p^{\min}-p_t\right]+\sum_{t=1}^{T}\mu_t^3\left[p_t-p^{\max}\right]+\sum_{t=1}^{T}\mu_t^4\left[p_t-p_{t-1}-r^{\text{up}}\right]$$
$$+\sum_{t=1}^{T}\mu_t^5\left[p_{t-1}-p_t-r^{\text{down}}\right] \tag{79}$$

where $\mathbf{p}$, $\boldsymbol{\mu}^2$ $\boldsymbol{\mu}^3$ $\boldsymbol{\mu}^4$ and $\boldsymbol{\mu}^5$ are vectors of primal and dual variables.

Then, the sensitivities with respect to the data, using the notation $L_1(\mathbf{p},\boldsymbol{\mu})$ for $L_1(\mathbf{p},\mu^1,\boldsymbol{\mu}^2,\boldsymbol{\mu}^3,\boldsymbol{\mu}^4,\boldsymbol{\mu}^5)$, are:

$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial \lambda_r^{\text{avg}}} = p_r; \qquad r=1,2,\ldots,T \tag{80}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial a} = -\frac{1}{2}\sum_{t=1}^{T}p_t^2 \tag{81}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial b} = -\sum_{t=1}^{T}p_t \tag{82}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial R} = -\mu^1 \tag{83}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial p^{\min}} = \sum_{t=1}^{T}\mu_t^2 \tag{84}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial p^{\max}} = -\sum_{t=1}^{T}\mu_t^3 \tag{85}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial r^{\text{up}}} = -\sum_{t=1}^{T}\mu_t^4 \tag{86}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial r^{\text{down}}} = -\sum_{t=1}^{T}\mu_t^5 \tag{87}$$
$$\frac{\partial L_1(\mathbf{p},\boldsymbol{\mu})}{\partial Q_{rs}^{\text{est}}} = \mu^1 p_r p_s; \qquad r,s=1,2,\ldots,T \tag{88}$$

Similarly, the Lagrange function associated with the problem (74)-(78) is

$$L_2(\mathbf{p},\rho^1,\boldsymbol{\rho}^2,\boldsymbol{\rho}^3,\boldsymbol{\rho}^4,\boldsymbol{\rho}^5) = \sum_{i=1}^{T}\sum_{j=1}^{T}Q_{ij}^{\text{est}}p_ip_j+\rho^1\left[B-\sum_{t=1}^{T}\left(\lambda_t^{\text{avg}}p_t-bp_t-\frac{1}{2}ap_t^2\right)\right]$$
$$\sum_{t=1}^{T}\rho_t^2\left[p^{\min}-p_t\right]+\sum_{t=1}^{T}\rho_t^3\left[p_t-p^{\max}\right]+\sum_{t=1}^{T}\rho_t^4\left[p_t-p_{t-1}-r^{\text{up}}\right]$$
$$+\sum_{t=1}^{T}\rho_t^5\left[p_{t-1}-p_t-r^{\text{down}}\right] \tag{89}$$

where $\mathbf{p}$, $\boldsymbol{\rho}^2$ $\boldsymbol{\rho}^3$ $\boldsymbol{\rho}^4$ and $\boldsymbol{\rho}^5$ are vectors of primal and dual variables.

Then, the sensitivities with respect to the data, using the notation $L_2(\mathbf{p}, \boldsymbol{\rho})$ for $L_2(\mathbf{p}, \rho^1, \boldsymbol{\rho}^2, \boldsymbol{\rho}^3, \boldsymbol{\rho}^4, \boldsymbol{\rho}^5)$, are:

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial \lambda_r^{\text{avg}}} = -\rho^1 p_r = -p_r/\mu^1; \qquad r = 1, 2, \ldots, T \tag{90}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial a} = \frac{\rho^1}{2}\sum_{t=1}^{T} p_t^2 = \frac{1}{2\mu^1}\sum_{t=1}^{T} p_t^2; \tag{91}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial b} = \rho^1 \sum_{t=1}^{T} p_t = \frac{1}{\mu^1}\sum_{t=1}^{T} p_t \tag{92}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial B} = \rho^1 = 1/\mu^1 \tag{93}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial p^{\min}} = \sum_{t=1}^{T} \rho_t^2 = \frac{1}{\mu^1}\sum_{t=1}^{T} \mu_t^2 \tag{94}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial p^{\max}} = -\sum_{t=1}^{T} \rho_t^3 = -\frac{1}{\mu^1}\sum_{t=1}^{T} \mu_t^3 \tag{95}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial r^{\text{up}}} = -\sum_{t=1}^{T} \rho_t^4 = -\frac{1}{\mu^1}\sum_{t=1}^{T} \mu_t^4 \tag{96}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial r^{\text{down}}} = -\sum_{t=1}^{T} \rho_t^5 = -\frac{1}{\mu^1}\sum_{t=1}^{T} \mu_t^5 \tag{97}$$

$$\frac{\partial L_2(\mathbf{p}, \boldsymbol{\rho})}{\partial Q_{rs}^{\text{est}}} = p_r p_s; \qquad r, s = 1, 2, \ldots, T \tag{98}$$

Note that equation (54) is used to express the above sensitivities as a function of dual variables $\boldsymbol{\mu}$

6.2 Numerical Example

Consider now a numerical example of the case study assuming that we are in an electric market, where

$$a = 0.070\ \$/(\text{MW})^2\text{h}; \quad b = 18\ \$/\text{MWh}; \quad R = 6000\ \$^2; \quad p^{\max} = 60\ \text{MW}$$

$$p^{\min} = 0\ \text{MW}; \quad r^{\text{up}} = 30\ \text{MW/h}; \quad r^{\text{down}} = 30\ \text{MW/h}; \quad p_0 = 0\ \text{MW}.$$

Table I: Price estimates (\$/MWh) as a function of time t.

λ_t^{est}							
t	λ_t	t	λ_t	t	λ_t	t	λ_t
1	33.31	7	24.65	13	41.05	19	40.74
2	26.53	8	24.75	14	41.61	20	38.80
3	22.16	9	25.50	15	38.98	21	39.63
4	23.10	10	27.58	16	39.74	22	46.14
5	22.60	11	31.60	17	42.02	23	39.04
6	23.15	12	35.60	18	42.09	24	33.68

and where the $\lambda_t; t = 1, 2, \ldots, T$ are given in Table I.

The Q_{ij}^{est} matrix is:

$$\begin{pmatrix}
1.60 & -0.40 & 0.40 & 0.14 & 0.10 & 0.09 & 0.10 & 0.58 & 0.27 & 0.40 & 0.09 & 0.10 & 0.27 & 0.19 & -0.14 & 0.08 & 0.23 & 0.06 & 0.19 & -0.16 & 0.02 & -0.15 & -0.19 & -0.01 \\
-0.40 & 0.37 & 0.08 & 0.00 & 0.05 & 0.05 & 0.06 & -0.10 & -0.05 & -0.05 & -0.12 & 0.01 & -0.05 & -0.08 & 0.04 & 0.05 & 0.02 & 0.17 & -0.05 & 0.07 & -0.20 & 0.02 & -0.15 & 0.10 \\
0.40 & 0.08 & 0.61 & 0.08 & 0.20 & 0.16 & 0.08 & 0.37 & 0.07 & 0.18 & -0.29 & 0.08 & 0.10 & 0.12 & 0.15 & 0.16 & 0.29 & 0.18 & 0.06 & 0.03 & -0.11 & -0.32 & -0.13 & -0.10 \\
0.14 & 0.00 & 0.08 & 0.32 & 0.09 & 0.15 & 0.11 & 0.10 & 0.06 & -0.01 & -0.07 & -0.05 & 0.09 & -0.06 & 0.09 & 0.08 & 0.08 & 0.02 & 0.01 & 0.00 & -0.04 & -0.02 & 0.06 & -0.05 \\
0.10 & 0.05 & 0.20 & 0.09 & 0.25 & 0.15 & 0.23 & 0.43 & 0.13 & 0.07 & -0.11 & -0.10 & 0.12 & -0.07 & 0.00 & 0.17 & 0.19 & 0.15 & 0.04 & 0.01 & -0.15 & -0.32 & -0.10 & -0.15 \\
0.09 & 0.05 & 0.16 & 0.15 & 0.15 & 0.19 & 0.31 & 0.32 & 0.18 & 0.10 & -0.11 & -0.05 & 0.08 & -0.06 & -0.01 & 0.14 & 0.13 & 0.12 & 0.04 & -0.04 & -0.17 & -0.28 & -0.12 & -0.12 \\
0.10 & 0.06 & 0.08 & 0.11 & 0.23 & 0.31 & 1.49 & 1.18 & 0.85 & 0.16 & -0.18 & -0.19 & -0.20 & -0.17 & -0.40 & 0.32 & 0.09 & 0.24 & 0.00 & -0.29 & -0.65 & -1.03 & -0.71 & -0.26 \\
0.58 & -0.10 & 0.37 & 0.10 & 0.43 & 0.32 & 1.18 & 1.76 & 0.83 & 0.55 & 0.01 & -0.11 & 0.09 & 0.09 & -0.38 & 0.43 & 0.39 & 0.27 & 0.07 & -0.14 & -0.56 & -1.03 & -0.59 & -0.44 \\
0.27 & -0.05 & 0.07 & 0.06 & 0.13 & 0.18 & 0.85 & 0.83 & 1.21 & 0.35 & 0.18 & 0.07 & 0.11 & -0.06 & -0.29 & 0.27 & 0.10 & 0.11 & 0.02 & -0.15 & -0.39 & -0.66 & -0.33 & -0.25 \\
0.40 & -0.05 & 0.18 & -0.01 & 0.07 & 0.10 & 0.16 & 0.55 & 0.35 & 0.72 & 0.27 & 0.24 & 0.24 & 0.17 & -0.20 & 0.12 & 0.15 & 0.11 & 0.08 & -0.01 & -0.20 & -0.05 & -0.23 & -0.15 \\
0.09 & -0.12 & -0.29 & -0.07 & -0.11 & -0.11 & -0.18 & 0.01 & 0.18 & 0.27 & 0.90 & 0.14 & 0.27 & 0.34 & -0.11 & -0.20 & 0.00 & -0.08 & 0.00 & 0.12 & 0.03 & 0.44 & 0.18 & 0.01 \\
0.10 & 0.01 & 0.08 & -0.05 & -0.10 & -0.05 & -0.19 & -0.11 & 0.07 & 0.24 & 0.14 & 0.42 & -0.03 & 0.25 & -0.06 & 0.01 & -0.05 & 0.00 & 0.05 & 0.09 & -0.07 & 0.33 & -0.09 & 0.20 \\
0.27 & -0.05 & 0.10 & 0.09 & 0.12 & 0.08 & -0.20 & 0.09 & 0.11 & 0.24 & 0.27 & -0.03 & 0.48 & -0.01 & 0.08 & 0.01 & 0.21 & 0.02 & 0.11 & 0.06 & 0.08 & 0.01 & 0.21 & -0.15 \\
0.19 & -0.08 & 0.12 & -0.06 & -0.07 & -0.06 & -0.17 & 0.09 & -0.06 & 0.17 & 0.34 & 0.25 & -0.01 & 0.60 & 0.06 & -0.04 & 0.03 & -0.01 & 0.02 & 0.15 & 0.06 & 0.20 & 0.12 & 0.17 \\
-0.14 & 0.04 & 0.15 & 0.09 & 0.00 & -0.01 & -0.40 & -0.38 & -0.29 & -0.20 & -0.11 & -0.06 & 0.08 & 0.06 & 0.43 & -0.07 & 0.11 & -0.13 & 0.04 & 0.13 & 0.29 & 0.20 & 0.40 & -0.05 \\
0.08 & 0.05 & 0.16 & 0.08 & 0.17 & 0.14 & 0.32 & 0.43 & 0.27 & 0.12 & -0.20 & 0.01 & 0.01 & -0.04 & -0.07 & 0.39 & 0.15 & 0.19 & 0.09 & 0.08 & -0.24 & -0.22 & -0.08 & -0.04 \\
0.23 & 0.02 & 0.29 & 0.08 & 0.19 & 0.13 & 0.09 & 0.39 & 0.10 & 0.15 & 0.00 & -0.05 & 0.21 & 0.03 & 0.11 & 0.15 & 0.31 & 0.07 & 0.11 & 0.08 & -0.05 & -0.12 & 0.05 & -0.22 \\
0.06 & 0.17 & 0.18 & 0.02 & 0.15 & 0.12 & 0.24 & 0.27 & 0.11 & 0.11 & -0.08 & 0.00 & 0.02 & -0.01 & -0.13 & 0.19 & 0.07 & 0.41 & -0.07 & 0.05 & -0.27 & -0.19 & -0.25 & 0.07 \\
0.19 & -0.05 & 0.06 & 0.01 & 0.04 & 0.04 & 0.00 & 0.07 & 0.02 & 0.08 & 0.00 & 0.05 & 0.11 & 0.02 & 0.04 & 0.09 & 0.11 & -0.07 & 0.21 & 0.02 & 0.00 & 0.01 & 0.04 & -0.01 \\
-0.16 & 0.07 & 0.03 & 0.00 & 0.01 & -0.04 & -0.29 & -0.14 & -0.15 & -0.01 & 0.12 & 0.09 & 0.06 & 0.15 & 0.13 & 0.08 & 0.08 & 0.05 & 0.02 & 0.24 & 0.03 & 0.32 & 0.23 & 0.09 \\
0.02 & -0.20 & -0.11 & -0.04 & -0.15 & -0.17 & -0.65 & -0.56 & -0.39 & -0.20 & 0.03 & -0.07 & 0.08 & 0.06 & 0.29 & -0.24 & -0.05 & -0.27 & 0.00 & 0.03 & 0.66 & 0.35 & 0.56 & -0.02 \\
-0.15 & 0.02 & -0.32 & -0.02 & -0.32 & -0.28 & -1.03 & -1.03 & -0.66 & -0.05 & 0.44 & 0.33 & 0.01 & 0.20 & 0.20 & -0.22 & -0.12 & -0.19 & 0.01 & 0.32 & 0.35 & 1.71 & 0.45 & 0.26 \\
-0.19 & -0.15 & -0.13 & 0.06 & -0.10 & -0.12 & -0.71 & -0.59 & -0.33 & -0.23 & 0.18 & -0.09 & 0.21 & 0.12 & 0.40 & -0.08 & 0.05 & -0.25 & 0.04 & 0.23 & 0.56 & 0.45 & 0.84 & -0.03 \\
-0.01 & 0.10 & -0.10 & -0.05 & -0.15 & -0.12 & -0.26 & -0.44 & -0.25 & -0.15 & 0.01 & 0.20 & -0.15 & 0.17 & -0.05 & -0.04 & -0.22 & 0.07 & -0.01 & 0.09 & -0.02 & 0.26 & -0.03 & 0.60
\end{pmatrix}$$

If we solve the problem (69)-(73) we obtain and optimal benefit of \$8,225 for the optimal productions ($p_t; t = 1, 2, \ldots, T$) indicated in the first column of Table II. The values of the associated dual variables are $\mu^1 = 0.5512$, and for $\boldsymbol{\mu}^2, \boldsymbol{\mu}^3, \boldsymbol{\mu}^4$ and $\boldsymbol{\mu}^5$ are the negative of the values in the columns 2 to 5 in Table II.

The sensitivities of the expected benefits with respect to the data values, calculated using the formulas (80)-(88), are:

$$\frac{\partial \text{Profit}}{\partial a} = -9063; \quad \frac{\partial \text{Profit}}{\partial b} = -468.6; \quad \frac{\partial \text{Profit}}{\partial R} = 0.55; \quad \frac{\partial \text{Profit}}{\partial p^{\min}} = 46.86;$$

$$\frac{\partial \text{Profit}}{\partial p^{\max}} = 2.31; \quad \frac{\partial \text{Profit}}{\partial r^{\text{up}}} = 15.17; \quad \frac{\partial \text{Profit}}{\partial r^{\text{down}}} = 12.7.$$

The sensitivities of the risk measure with respect to the data values, calculated using the formulas (90)-(98), are:

$$\frac{\partial \text{Risk}}{\partial a} = 16442; \quad \frac{\partial \text{Risk}}{\partial b} = -850.20; \quad \frac{\partial \text{Risk}}{\partial B} = -1.81; \quad \frac{\partial \text{Risk}}{\partial p^{\min}} = -85.01;$$

$$\frac{\partial \text{Risk}}{\partial p^{\max}} = -4.19; \quad \frac{\partial \text{Risk}}{\partial r^{\text{up}}} = -27.53; \quad \frac{\partial \text{Risk}}{\partial r^{\text{down}}} = -23.10.$$

Alternatively, the sensitivities of the uncertainties with respect to r^{up}, r^{dowm}, $p^{\min}$ and $p^{\max}$ could be calculated solving the problem (74)-(78) that leads to the same optimal productions $(p_t; t = 1, 2, \ldots, T)$ and the associated dual variables are $\rho^1 = 1/\mu^1 = 1/0.5512 = 1.8141$, and for $\boldsymbol{\rho}^2, \boldsymbol{\rho}^3, \boldsymbol{\rho}^4$ and $\boldsymbol{\rho}^5$ are the negative of the values in the columns 6 to 9 in Table II.

Note that the theory developed in Section 5 applies here.

7 Conclusions

In this paper a general method for sensitivity analysis, which is applicable to any model that can be formulated as an optimization problem, has been introduced. It has been shown that by considering the data as artificial variables and setting them to their actual values, the sensitivity with respect to any parameter can be obtained. More precisely, Theorem 2 provides a powerful tool to derive analytical expressions for the sensitivities. In addition, not only the sensitivities of the objective function but the sensitivities of any active constraint with respect to all parameters can be calculated without extra computational requirements, because the solution of the primal problem together with the values of the dual variables are sufficient to derive these sensitivities. So, the proposed method for calculation of the local sensitivities is computationally inexpensive. In addition, if the local sensitivities are direction dependent (degenerated cases), the method remains valid and allows determining the corresponding right and left derivatives. The power of the method has been proved and illustrated by its application to two examples, one degenerated linear problem and one related to a competitive market.

Table II: Productions and sensitivities of the expected benefits and uncertainties with respect to r^{up}, r^{down}, $P^{\min}$ and $P^{\max}$.

Time	Productions	Benefit Sensitivities to				Risk Sensitivities to			
		r^{up}	r^{dowm}	$p^{\min}$	$p^{\max}$	r^{up}	r^{dowm}	$p^{\min}$	$p^{\max}$
t	p_t	$-\mu_t^4$	$-\mu_t^5$	$-\mu_t^2$	$-\mu_t^3$	$-\rho_t^4$	$-\rho_t^5$	$-\rho_t^2$	$-\rho_t^3$
1	0.00	0.00	0.00	-9.96	0.00	0.00	0.00	18.08	0.00
2	13.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	-17.88	0.00	0.00	0.00	32.43	0.00
4	0.00	0.00	0.00	-7.07	0.00	0.00	0.00	12.82	0.00
5	0.00	0.00	0.00	-5.73	0.00	0.00	0.00	10.39	0.00
6	12.94	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	42.94	5.59	0.00	0.00	0.00	-10.14	0.00	0.00	0.00
8	12.94	0.00	5.12	0.00	0.00	0.00	-9.30	0.00	0.00
9	11.27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	-6.14	0.00	0.00	0.00	11.14	0.00
11	0.19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	21.64	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
13	17.11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	-0.08	0.00	0.00	0.00	0.15	0.00
15	30.00	0.39	0.00	0.00	0.00	-0.70	0.00	0.00	0.00
16	0.00	0.00	1.85	0.00	0.00	0.00	-3.36	0.00	0.00
17	24.85	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	54.85	3.28	0.00	0.00	0.00	-5.94	0.00	0.00	0.00
19	57.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	30.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	60.00	5.16	0.00	0.00	2.31	-9.36	0.00	0.00	-4.19
22	30.00	0.00	5.76	0.00	0.00	0.00	-10.44	0.00	0.00
23	9.76	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	39.76	0.76	0.00	0.00	0.00	-1.38	0.00	0.00	0.00

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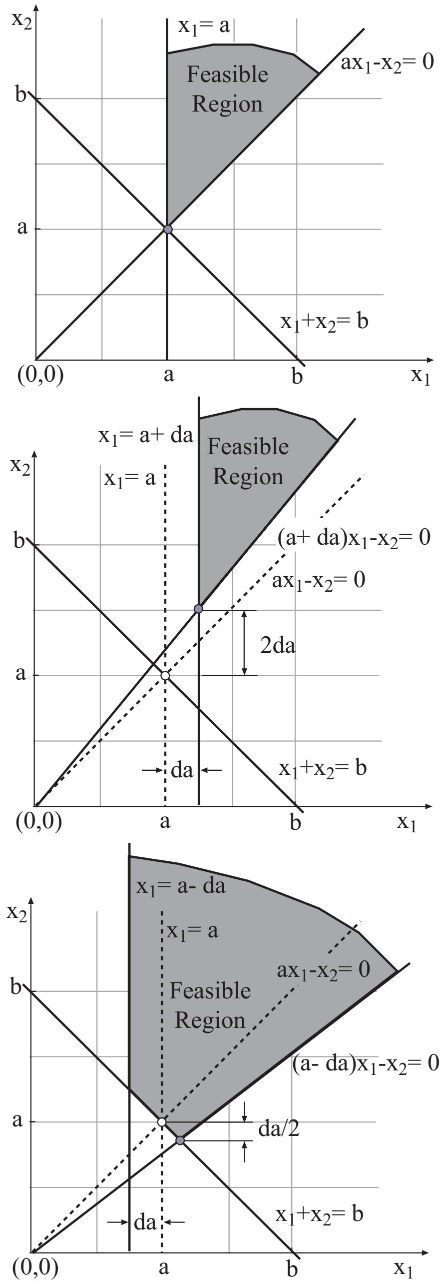

Figure 1: Illustration of the feasible regions and optimal values of the initial and modified problems due to changes in the a parameter. (Upper) Initial problem. (Middle) Positive increment of a. (Lower) Negative increment of a.

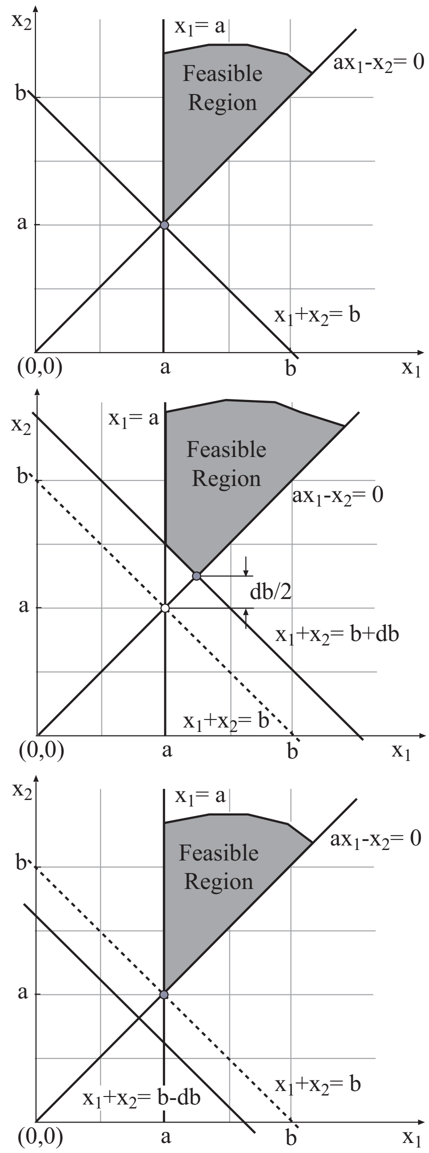

Figure 2: Illustration of the feasible regions and optimal values of the initial and modified problems due to changes in the b parameter. (Upper) Initial problem. (Middle) Positive increment of b. (Lower) Negative increment of b.