Closed Formulas in Local Sensitivity Analysis for Some Classes of Linear and Non-Linear Problems

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Abstract

This paper presents an integrated approach to sensitivity analysis in some linear and non-linear programming problems. Closed formulas for the sensitivities of the objective function and primal and dual variables with respect to all parameters for some classes of problems are obtained. As particular cases, the sensitivities with respect to all data values, i.e., cost coefficients, constraints coefficients and right hand side terms of the constraints are provided for these classes of problems as closed formulas. The method is illustrated by its application to several examples.

Key Words: Local sensitivity, mathematical programming, duality, closed formulas for sensitivities.

1 Introduction

This paper deals with a long familiar problem: sensitivity analysis in linear and non-linear programming. Sensitivity analysis consists of determining "how" and "how much" specific changes in the parameters of an optimization problem modify the optimal objective function value and the point where the optimum is attained.

Sensitivity analysis should be a routine complement, which adds quality to solving methods. Today, users are not satisfied with the solutions to given problems and they require information of how these solutions depend on data.

The problem of sensitivity analysis in linear and nonlinear programming has been discussed by several authors, as, for example, Fiacco [9, 10], Fiacco and McCormick [11], Vanderplaats [18], Sobiesky et al. [16], Enevoldsen [8], Gauvin [12], Bonnans and Shapiro [2, 3], Castillo et al. [4, 6], Klatte and Kummer [13], etc. There are at least three ways of deriving equations for the unknown sensitivities: (a) the Lagrange multiplier equations of the constrained optimum (see Sobiesky et al. [16]), (b) differentiation of the Karush–Kuhn–Tucker conditions to obtain the sensitivities of the objective function with respect to changes in the parameters (see Sorensen and Enevoldsen [17] or Enevoldsen, [8]), and (c) the extreme conditions of a penalty function (see Sobiesky et al. [16]).

The existing methods for sensitivity analysis may present three main limitations:

1. Most of them provide the sensitivities of the objective function value and the primal variables values with respect to some parameters, but not the sensitivities of the dual variables with respect to all parameters.

2. Each of the sensitivities (optimal objective function value or primal variable values with respect to parameters) is obtained using a different approach, but there is no integrated approach providing all the sensitivities at once. This is the case of the methods in the available literature that use specific techniques to attain different sensitivities. Furthermore, in textbooks, results are frequently reported in a disperse (often confuse) and non-general manner.

3. Some sensitivity formulas are given in terms of some long expressions involving the inverses of partitioned matrices (see for example Fiacco and McCormick [11], page 34–36).

4. There are not formulas for the sensitivities with respect to parameters which appear in several places (objective function, constraint coefficients and/or in the right hand side of the constraints).

The aim of this paper, which is limited to some wide classes of problems, is twofold: (a) to perform an integrated sensitivity analysis in which all the sensitivities (objective function value, primal and dual variables values with respect to the parameters) are obtained at once, and (b) to give closed and simple formulas for the sensitivities in the general case, i.e., when the cost coefficients, right hand side parameters and constraint matrix coefficients depend on parameters. As particular cases they include the sensitivities of: (a) the objective function, (b) all primal variables and (c) all dual variables with respect to: (i) cost coefficients, (ii) right hand side parameters, and (iii) constraint coefficients.

Though some of the specific sensitivity results are certainly not novel, the resulting general and compact algebraic formulas certainly are. In particular, the set of sensitivity formulas provided in (38) and (49) for the case of linear programming provide a very neat result.

Since the sensitivities studied in the paper are local sensitivities, that is, changes produced by differential changes, if the problem is regular and non-degenerated, the set of active constraints remains unchanged. Finally, it should be noted that the compact formulas derived are only valid for the case of unique solution and non-redundant constraints, as indicated.

This paper is structured as follows. In Section 2 the necessary background to deal with this problem is given. In Section 3 a method for determining the set of all feasible perturbations for linear programming problems is provided. In Section 4 the formulas are applied to the case of non-linear programming. All the formulas are illustrated by means of examples. Finally, in Section 5 some conclusions are given.

2 Sensitivity analysis

Consider the following Nonlinear Programming Problem (NLPP):

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad z = f(\boldsymbol{x}, \boldsymbol{\theta}) \tag{1}$$

subject to

$$\begin{array}{rcl} \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) & = & \boldsymbol{0} \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta}) & \leq & \boldsymbol{0}, \end{array} \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, $\boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^\ell$, $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ with $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) = (h_1(\boldsymbol{x}, \boldsymbol{\theta}), \ldots, h_\ell(\boldsymbol{x}, \boldsymbol{\theta}))^T$ and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta}) = (g_1(\boldsymbol{x}, \boldsymbol{\theta}), \ldots, g_m(\boldsymbol{x}, \boldsymbol{\theta}))^T$ are functions over the feasible region $S(\boldsymbol{\theta}) = \{\boldsymbol{x} | \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta}) \leq \boldsymbol{0}\}$ and $f, \boldsymbol{h}, \boldsymbol{g} \in C^2$. We assume that $\ell \leq n$.

Let $\boldsymbol{x}^*$ be an optimal solution of problem (1)-(2) and assume that:

1. Point $\boldsymbol{x}^*$ is a regular point of the constraints $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\theta}) = \boldsymbol{0}$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}) \le \boldsymbol{0}$, i.e., if J is the set of indices j for which $g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) = 0$, the gradient vectors $\nabla_{\boldsymbol{x}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta})$, $\nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta})$, $k = 1,\dots,\ell; j \in J$ are linearly independent, that is, they satisfy the linear independent constraint qualification (LICQ) (note that other regularization conditions are possible, see Luenberger [14]).

2. Point $\boldsymbol{x}^*$ is a non-degenerated point, i.e., all dual variables associated with the active inequalities are non-null (this property is also called strict complementarity).

To derive the sensitivity formulas, we consider the Karush–Kuhn–Tucker (KKT) (see [1] or [14]) first order optimality conditions for problem (1)-(2):

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{x}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{j=1}^{m} \mu_j^* \nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) = \boldsymbol{0} \quad (3)$$

$$h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) = 0; \ \ k = 1,2,\dots,\ell \quad (4)$$

$$g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) \le 0; \ \ j = 1,2,\dots,m \quad (5)$$

$$\mu_j^* \, g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) = 0; \ \ j = 1,2,\dots,m \quad (6)$$

$$\mu_j^* \ge 0; \ \ j = 1,2,\dots,m. \quad (7)$$

To obtain sensitivity equations, we perturb or modify $\boldsymbol{x}^*$, $\boldsymbol{\theta}$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, z^* in such a way that the KKT conditions still hold. Thus, to obtain the sensitivity equations we differentiate the objective function (1) and the optimality conditions (3)-(7), as follows:

$$\begin{aligned}
[\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{x} + [\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{\theta} - dz \;&=\; 0 && (8)\\
\left[\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{j=1}^{|J|} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta})\right] d\boldsymbol{x} \;+& \\
\left[\nabla_{\boldsymbol{x\theta}} f(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{x\theta}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{j=1}^{|J|} \mu_j^* \nabla_{\boldsymbol{x\theta}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta})\right] d\boldsymbol{\theta} \;+& \\
\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{\theta}) d\boldsymbol{\lambda} + \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{\theta}) d\boldsymbol{\mu} \;&=\; \boldsymbol{0} && (9)\\
[\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{x} + [\nabla_{\boldsymbol{\theta}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{\theta} \;&=\; \boldsymbol{0} && (10)\\
[\nabla_{\boldsymbol{x}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{x} + [\nabla_{\boldsymbol{\theta}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta})]^T d\boldsymbol{\theta} \;&=\; 0 \\
\text{if } \mu_j^* \;&\ne\; 0; j \in J, && (11)
\end{aligned}$$

where J is the set of binding (active) inequality constraints, $|J|$ its cardinality, and all the matrices are evaluated at the optimal solution, $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, z^*.

Note that the differentiated equation (6) is not present in the previous list because it holds due to the fact that we consider only non-degenerated cases (a detailed explanation can be seen in Castillo e al. [6], pages 53 and 54).

Observe that the uncommon degenerated case (binding inequality constraints with null multipliers) is not considered here. The degenerate case is analyzed in [2, 3] and [6], pages 61 to 72.

The linear system of equations (8)-(11) can be expressed in matrix form as follows

$$\left[\begin{array}{c|c|c|c|c} \boldsymbol{F_x} & \boldsymbol{F_\theta} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{F_{x\theta}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{H_\theta} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x} & \boldsymbol{G_\theta} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right] \left[\begin{array}{c} d\boldsymbol{x} \\ d\boldsymbol{\theta} \\ d\boldsymbol{\lambda} \\ d\boldsymbol{\mu} \\ dz \end{array}\right] = \boldsymbol{0} \tag{12}$$

where the vectors and submatrices in (12) are defined below (dimensions in parenthesis)

$$\boldsymbol{F_x}_{(1\times n)} = [\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \tag{13}$$

$$\boldsymbol{F_\theta}_{(1\times p)} = [\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \tag{14}$$

$$\boldsymbol{F_{xx}}_{(n\times n)} = \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{xx}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{j=1}^{|J|} \mu_j^* \nabla_{\boldsymbol{xx}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) \tag{15}$$

$$\boldsymbol{F_{x\theta}}_{(n\times p)} = \nabla_{\boldsymbol{x\theta}} f(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{k=1}^{\ell} \lambda_k^* \nabla_{\boldsymbol{x\theta}} h_k(\boldsymbol{x}^*,\boldsymbol{\theta}) + \sum_{j=1}^{|J|} \mu_j^* \nabla_{\boldsymbol{x\theta}} g_j(\boldsymbol{x}^*,\boldsymbol{\theta}) \tag{16}$$

$$\boldsymbol{H_x}_{(\ell\times n)} = [\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \tag{17}$$

$$\boldsymbol{H_\theta}_{(\ell\times p)} = [\nabla_{\boldsymbol{\theta}} \boldsymbol{h}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \tag{18}$$

$$\boldsymbol{G_x}_{(|J|\times n)} = [\nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \tag{19}$$

$$\boldsymbol{G_\theta}_{(|J|\times p)} = [\nabla_{\boldsymbol{\theta}} \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{\theta})]^T \ . \tag{20}$$

Vector (13) is the gradient of the objective function with respect to $\boldsymbol{x}$, vector (14) is the gradient on the objective function with respect to $\boldsymbol{\theta}$, submatrix (15) is the Hessian of the Lagrangian $(f(\boldsymbol{x},\boldsymbol{\theta})+\boldsymbol{\lambda}^T\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\theta})+\boldsymbol{\mu}^T\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}))$ with respect to $\boldsymbol{x}$, submatrix (16) is the Hessian of the Lagrangian with respect to $\boldsymbol{x}$ and $\boldsymbol{\theta}$, submatrix (17) is the Jacobian of $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\theta})$ with respect to $\boldsymbol{x}$, submatrix (18) is the Jacobian of $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$, submatrix (19) is the Jacobian of $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta})$ with respect to $\boldsymbol{x}$ for binding constraints, and submatrix (20) is the Jacobian of $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ for binding constraints.

To compute all sensitivities with respect to the components of the parameter vector $\boldsymbol{\theta}$, the system (12) can be written as

$$\boldsymbol{U}\,[\ d\boldsymbol{x} \quad d\boldsymbol{\lambda} \quad d\boldsymbol{\mu} \quad dz\ ]^T = \boldsymbol{S}d\boldsymbol{\theta}, \tag{21}$$

where the matrices $\boldsymbol{U}$ and $\boldsymbol{S}$ are

$$\boldsymbol{U} = \left[\begin{array}{c|c|c|c} \boldsymbol{F_x} & \boldsymbol{0} & \boldsymbol{0} & -1 \\ \hline \boldsymbol{F_{xx}} & \boldsymbol{H_x^T} & \boldsymbol{G_x^T} & 0 \\ \hline \boldsymbol{H_x} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \hline \boldsymbol{G_x} & \boldsymbol{0} & \boldsymbol{0} & 0 \end{array}\right]; \quad \boldsymbol{S} = -\left[\begin{array}{c} \boldsymbol{F_\theta} \\ \boldsymbol{F_{x\theta}} \\ \boldsymbol{H_\theta} \\ \boldsymbol{G_\theta} \end{array}\right] \tag{22}$$

and therefore, if $\boldsymbol{U}$ is invertible, we have

$$\left[\ d\boldsymbol{x} \quad d\boldsymbol{\lambda} \quad d\boldsymbol{\mu} \quad dz\ \right]^T = \left[\ \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} \quad \frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} \quad \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}} \quad \frac{\partial z}{\partial \boldsymbol{\theta}}\ \right]^T d\boldsymbol{\theta} = \boldsymbol{U}^{-1}\boldsymbol{S}\, d\boldsymbol{\theta}, \tag{23}$$

from which the matrix of all partial derivatives with respect to parameters results

$$\left[\begin{array}{cccc}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}} & \frac{\partial z}{\partial \boldsymbol{\theta}}\end{array}\right]^T = \boldsymbol{U}^{-1}\boldsymbol{S}\,. \tag{24}$$

Expression (24) allows one deriving sensitivities of the variables, the multipliers (dual variables) and the objective function with respect to all parameters.

Two particular but very interesting cases are the following:

Case 1: $\ell + |J| = n$, i.e., when the number of active constraints (equalities plus inequalities) coincides with the number of variables, and the matrix $\boldsymbol{Q}$ is invertible.

Case 2: $\boldsymbol{F_{xx}}$ is positive definite (invertible) and $\boldsymbol{Q} = \begin{pmatrix}\boldsymbol{H}\\ \boldsymbol{G}\end{pmatrix}$ is full row rank (a typical assumption).

Note that in these two cases, the invertibility of $\boldsymbol{U}$ is guaranteed, and that formula (24) is valid for all cases in which $\boldsymbol{U}$ is invertible.

In the first case, if we denote $\boldsymbol{Q} = \begin{pmatrix}\boldsymbol{H}\\ \boldsymbol{G}\end{pmatrix}$ and $\boldsymbol{\eta} = \begin{pmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{pmatrix}$, matrix $\boldsymbol{U}$ can be written as

$$\boldsymbol{U} = \left[\begin{array}{c|c|c}\boldsymbol{F_x} & \boldsymbol{0} & -1\\ \hline \boldsymbol{F_{xx}} & \boldsymbol{Q_x^T} & 0\\ \hline \boldsymbol{Q_x} & \boldsymbol{0} & 0\end{array}\right]; \quad \boldsymbol{S} = -\left[\begin{array}{c}\boldsymbol{F_\theta}\\ \boldsymbol{F_{x\theta}}\\ \boldsymbol{Q_\theta}\end{array}\right], \tag{25}$$

and then, since $\boldsymbol{Q_x^{-1}}$ exists, its inverse also exists and can be written as

$$\boldsymbol{U}^{-1} = \left[\begin{array}{c|c|c}\boldsymbol{0}_{n\times 1} & \boldsymbol{0}_{n\times n} & \boldsymbol{Q_x}^{-1}_{n\times\ell}\\ \hline \boldsymbol{0}_{\ell\times 1} & (\boldsymbol{Q_x^T})^{-1}_{\ell\times n} & -(\boldsymbol{Q_x^T})^{-1}\boldsymbol{F_{xx}}\boldsymbol{Q_x}^{-1}_{\ell\times\ell}\\ \hline -1_{1\times 1} & \boldsymbol{0}_{1\times n} & \boldsymbol{F_x}\boldsymbol{Q_x}^{-1}_{1\times\ell}\end{array}\right], \tag{26}$$

leading to

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} = -\boldsymbol{Q_x^{-1}}\boldsymbol{Q_\theta} \tag{27}$$

$$\frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} = (\boldsymbol{Q_x^T})^{-1}\left(\boldsymbol{F_{xx}}\boldsymbol{Q_x^{-1}}\boldsymbol{Q_\theta} - \boldsymbol{F_{x\theta}}\right) \tag{28}$$

$$\frac{\partial Z}{\partial \boldsymbol{\theta}} = \boldsymbol{F_\theta} - \boldsymbol{F_x}\boldsymbol{Q_x^{-1}}\boldsymbol{Q_\theta} = \boldsymbol{F_\theta} + \boldsymbol{\lambda}^T\boldsymbol{Q_\theta}, \tag{29}$$

which are very neat formulas for the sensitivities.

The second case of invertible $\boldsymbol{F_{xx}}$ is dealt with in Section 4.

The importance of these formulas is that they can be applied to obtain at once all sensitivities without the need of discussing each sensitivity (objective functions, primal and dual variables) separately as it has been done in the past. These formulas are applied to the cases of linear and non-linear programming in the following sections, where compact formulas are obtained.

3 The Linear Programming Problem

Consider the following LP problem:

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad Z = \boldsymbol{c}^T \boldsymbol{x} \tag{30}$$

subject to

$$\begin{aligned} \boldsymbol{H}\boldsymbol{x} &= \boldsymbol{b}_H & (31)\\ \boldsymbol{G}\boldsymbol{x} &\leq \boldsymbol{b}_G & (32) \end{aligned}$$

where $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, $\boldsymbol{b}_H = (b_{H1}, b_{H2}, \dots, b_{H\ell})$, $\boldsymbol{b}_G = (b_{G1}, b_{G2}, \dots, b_{Gm})$, and $\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{G} \end{pmatrix}$ is a matrix of dimensions $(m+\ell) \times n$ with elements $q_{ij}; i = 1, 2, \dots m+\ell; j = 1, 2, \dots, n$.

3.1 Sensitivities with respect to data

In this section we are interested in determining the sensitivities of Z^*, $\boldsymbol{x}^*$ and the dual variables with respect to all the data. To this end one builds the matrix $\boldsymbol{A}$ of dimensions $n \times n$ including all the equalities in (31) and the active inequalities in (32), and the corresponding column matrix $\boldsymbol{b}$ including matrix $\boldsymbol{b}_H$ and the submatrix of $\boldsymbol{b}_G$ associated with the active inequality constraints (this is easy to achieve, especially if one has already solved the linear programming problem (30)-(32) by removing the non-active constraints). Note that the problem (30)-(32) is equivalent to minimize (30) subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{33}$$

Assuming that $m + \ell = n$, and that $\boldsymbol{A}$ is invertible, which is an important assumption that normally holds, one has the data

$$\boldsymbol{\theta} = (c_1, \dots, c_n; a_{11}, \dots, a_{n1}, a_{12}, \dots, a_{n2}, \dots, a_{1n}, \dots, a_{nn}, b_1, \dots, b_n).$$

Then, the matrix $\boldsymbol{U}$ becomes

$$\boldsymbol{U} = \left(\begin{array}{c|c|c} \boldsymbol{c}^T_{1\times n} & \boldsymbol{0}_{1\times n} & -1_{1\times 1} \\ \hline \boldsymbol{0}_{n\times n} & \boldsymbol{A}^T_{n\times n} & \boldsymbol{0}_{n\times 1} \\ \hline \boldsymbol{A}_{n\times n} & \boldsymbol{0}_{n\times n} & \boldsymbol{0}_{n\times 1} \end{array} \right) \tag{34}$$

whose symbolic inverse is

$$\boldsymbol{U}^{-1} = \left(\begin{array}{c|c|c} \boldsymbol{0}_{n\times 1} & \boldsymbol{0}_{n\times n} & \boldsymbol{A}^{-1}_{n\times n} \\ \hline \boldsymbol{0}_{n\times 1} & \left(\boldsymbol{A}^T\right)^{-1}_{n\times n} & \boldsymbol{0}_{n\times n} \\ \hline -1_{1\times 1} & \boldsymbol{0}_{1\times n} & \left(\boldsymbol{c}^T \boldsymbol{A}^{-1}\right)_{1\times n} \end{array} \right). \tag{35}$$

Note that if $\boldsymbol{A}$ is not invertible, the inverse of $\boldsymbol{U}$ does not exist.

The matrix $\boldsymbol{S}$ in this case is

$$\boldsymbol{S} = -\left(\begin{array}{c|c|c} \boldsymbol{x}^T_{1\times n} & \boldsymbol{0}_{1\times n^2} & \boldsymbol{0}_{1\times n} \\ \hline \boldsymbol{I}_n & \left(\boldsymbol{I}_n \otimes \boldsymbol{\lambda}^T\right)_{n\times n^2} & \boldsymbol{0}_{n\times n} \\ \hline \boldsymbol{0}_{n\times n} & \left(\boldsymbol{x}^T \otimes \boldsymbol{I}_n\right)_{n\times n^2} & -\boldsymbol{I}_n \end{array}\right), \tag{36}$$

where $\otimes$ refers to the tensor or Kronecker's product of matrices, which is defined as follows (see Ruiz-Tolosa and Castillo [15]).

Given two matrices

$$A_{m,n} \equiv [a_{\alpha\beta}]; \quad B_{p,q} \equiv [b_{\gamma\delta}]$$

of the indicated dimensions and elements, the matrix $P_{mp,nq} \equiv [p_{ij}]$ is said the Kronecker, direct or tensor product of matrices A and B and is denoted $P = A \otimes B$, if and only if

$$P_{mp,nq} = \left[\begin{array}{c|c|c|c} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \hline a_{21}B & a_{22}B & \dots & a_{2n}B \\ \hline \dots & \dots & \dots & \dots \\ \hline a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{array}\right],$$

where each partition has p rows and q columns.

Then, from (24) and using (35) and (36) the local sensitivities become

$$\begin{bmatrix} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} \\ \dfrac{\partial Z}{\partial \boldsymbol{\theta}} \end{bmatrix} = \boldsymbol{U}^{-1}\boldsymbol{S} = \left(\begin{array}{c|c|c} \boldsymbol{0} & -\left(\boldsymbol{b}^T\left(\boldsymbol{A}^T\right)^{-1}\right) \otimes \boldsymbol{A}^{-1} & \boldsymbol{A}^{-1} \\ \hline -\left(\boldsymbol{A}^T\right)^{-1} & -\left(\boldsymbol{A}^T\right)^{-1} \otimes \left(\boldsymbol{c}^T\boldsymbol{A}^{-1}\right) & \boldsymbol{0} \\ \hline \boldsymbol{b}^T\left(\boldsymbol{A}^T\right)^{-1} & -\left(\boldsymbol{b}^T\left(\boldsymbol{A}^T\right)^{-1}\right) \otimes \left(\boldsymbol{c}^T\boldsymbol{A}^{-1}\right) & \boldsymbol{c}^T\boldsymbol{A}^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} \boldsymbol{0} & -\boldsymbol{x}^T \otimes \boldsymbol{A}^{-1} & \boldsymbol{A}^{-1} \\ \hline -\left(\boldsymbol{A}^T\right)^{-1} & -\left(\boldsymbol{A}^T\right)^{-1} \otimes \boldsymbol{\lambda}^T & \boldsymbol{0} \\ \hline \boldsymbol{x}^T & \boldsymbol{x}^T \otimes \boldsymbol{\lambda}^T & -\boldsymbol{\lambda}^T \end{array}\right), \tag{37}$$

where the former matrix gives the sensitivities in terms of the data of the problem $\boldsymbol{A}, \boldsymbol{b}$ and $\boldsymbol{c}$, and the later, in terms of the primal variables $\boldsymbol{x}$, the dual variables $\boldsymbol{\lambda}$ and the matrix $\boldsymbol{A}$.

More precisely:

$$\begin{array}{llllll} \dfrac{\partial x_j}{\partial c_k} & = 0; & \dfrac{\partial x_j}{\partial a_{ik}} & = -a^{ji}x_k & \dfrac{\partial x_j}{\partial b_i} & = a^{ji} \\ \dfrac{\partial \lambda_i}{\partial c_j} & = -a^{ji}; & \dfrac{\partial \lambda_i}{\partial a_{\ell j}} & = -a^{ji}\lambda_\ell; & \dfrac{\partial \lambda_i}{\partial b_\ell} & = 0 \\ \dfrac{\partial Z}{\partial c_j} & = x_j; & \dfrac{\partial Z}{\partial a_{ij}} & = \lambda_i x_j; & \dfrac{\partial Z}{\partial b_i} & = -\lambda_i, \end{array} \tag{38}$$

where a^{ji} are the elements of $\boldsymbol{A}^{-1}$, which are directly supplied by the simplex method.

Observe that the equations (38) provide closed and very simple formulas for the sensitivities of the primal variables, the dual variables and the objective function with respect to all parameters in (30)-(32) when the number of active non-redundant constraints (equalities and active inequalities) is equal to the number of variables and $\boldsymbol{A}$ is invertible.

3.1.1 The diet problem

To illustrate the above results, consider the diet problem in Murty (1983). Let x_1, x_2 and x_3 be the amounts of greens, potatoes and corn (foods) included in the diet, respectively. The amounts of vitamins A, C and D, respectively in each food and the minimum daily requirements are given in Table 1.

Then, the well known diet problem, consisting of determining the diet (x_1, x_2 and x_3) such that the daily requirements are satisfied at minimum cost, becomes

$$\underset{x_1, x_2, x_3}{\text{Minimize}} \quad Z = 50x_1 + 100x_2 + 51x_3 \tag{39}$$

subject to

$$\begin{aligned}
10x_1 + x_2 + 9x_3 &\geq 5 && (40)\\
10x_1 + 10x_2 + 10x_3 &\geq 50 && (41)\\
10x_1 + 11x_2 + 11x_3 &\geq 10 && (42)\\
x_1 &\geq 0 && (43)\\
x_2 &\geq 0 && (44)\\
x_3 &\geq 0 && (45)
\end{aligned}$$

The optimal solution of problem (39)-(45) is

$$Z^* = 250; \quad x_1 = 5; \quad x_2 = 0; \quad x_3 = 0,$$

and the active constraints are (41), (44) and (45), with associated dual variables $\lambda_1 = -5, \quad \lambda_2 = -50, \quad \lambda_3 = -1$, respectively.

The resulting matrices $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are:

$$\boldsymbol{A} = \begin{pmatrix} 10 & 10 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \boldsymbol{A}^{-1} = \begin{pmatrix} 1/10 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Some interesting sensitivity questions are:

1. How does the optimal cost change when the amounts of vitamins A, B and C in each food, the daily vitamin requirements or the cost of food slightly change?
2. How does the optimal diet (x_1, x_2 and x_3) change when the amount of vitamins A, B and C in each food, the daily vitamin requirements or the cost of food slightly change?
3. How do the dual variables (λ_1, λ_2 and λ_3) change when the amount of vitamins A, B and C in each food, the daily vitamin requirements or the cost of food slightly change?

Table 1: Amounts of vitamins A, C and D in each food available and the minimum daily requirements.

	Food			Daily Requirement
	Greens	Potatoes	Corn	
Vitamin A	10	1	9	5
Vitamin C	10	10	10	50
Vitamin D	10	11	11	10
Cost ($/Kg)	50	100	51	

Table 2: Sensitivities with respect to a_{ij}, b_j and c_i for the diet example.

		Greens	Potatoes	Corn	Vitamin C	NN Potatoes	NN Corn	Cost
	a	$\frac{\partial x_1}{\partial a_{ij}}$	$\frac{\partial x_2}{\partial a_{ij}}$	$\frac{\partial x_3}{\partial a_{ij}}$	$\frac{\partial \lambda_1}{\partial a_{ij}}$	$\frac{\partial \lambda_2}{\partial a_{ij}}$	$\frac{\partial \lambda_3}{\partial a_{ij}}$	$\frac{\partial Z}{\partial a_{ij}}$
Vitamin C	a_{11}	-0.5	0	0	0.5	-5	-5	-25
	a_{12}	0	0	0	0	5	0	0
	a_{13}	0	0	0	0	0	5	0
NN Potatoes	a_{21}	5	-5	0	5	-50	-50	-250
	a_{22}	0	0	0	0	50	0	0
	a_{23}	0	0	0	0	0	50	0
NN Corn	a_{31}	5	0	-5	0.1	-1	-1	-5
	a_{32}	0	0	0	0	1	0	0
	a_{33}	0	0	0	0	0	1	0
	b	$\frac{\partial x_1}{\partial b}$	$\frac{\partial x_2}{\partial b}$	$\frac{\partial x_3}{\partial b}$	$\frac{\partial \lambda_1}{\partial b}$	$\frac{\partial \lambda_2}{\partial b}$	$\frac{\partial \lambda_3}{\partial b}$	$\frac{\partial Z}{\partial b}$
Vitamin C	b_1	0.1	0	0	0	0	0	5
NN Potatoes	b_2	-1	1	0	0	0	0	50
NN Corn	b_3	-1	0	1	0	0	0	1
	c	$\frac{\partial x_1}{\partial c}$	$\frac{\partial x_2}{\partial c}$	$\frac{\partial x_3}{\partial c}$	$\frac{\partial \lambda_1}{\partial c}$	$\frac{\partial \lambda_2}{\partial c}$	$\frac{\partial \lambda_3}{\partial c}$	$\frac{\partial Z}{\partial c}$
Vitamin C	c_1	0	0	0	-0.1	1	1	5
NN Potatoes	c_2	0	0	0	0	-1	0	0
NN Corn	c_3	0	0	0	0	0	-1	0

NN = non-negative.

Using Formulas (38) these questions can be answered. The sensitivities are provided in Table 2. The remaining sensitivities are null.

From Table 2 the following conclusions can be obtained:

1. The sensitivity $\frac{\partial Z}{\partial a_{11}} = -25$ shows the importance of increasing the amount of Vitamin C in greens, that would lead to an important decrease in cost.

2. Since only greens are used in the optimal solution, the only relevant non-null sensitivities are those related to greens data.

3. The sensitivities of the objective function with respect to the b's (see Table 2) are the dual variables changed sign, as expected.

4. All the sensitivities related to non-negative constraints have only sense from a mathematical point of view, because the non-negativity constraints cannot be changed.

5. All sensitivities of the optimal amounts of potatoes and corn with respect to any of the parameters (apart from the non-negativity ones) are null because the primal variables are null.

6. All sensitivities of the primal variables with respect to the cost coefficients c and those of the dual variables with respect to the b parameters are null, and this condition holds no matter which data values are chosen (see (38)).

3.2 Sensitivity with respect to arbitrary parameters

Assume now that all the data are functions of a set of parameters $\boldsymbol{\theta}$, that is:

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad Z = \boldsymbol{c}^T(\boldsymbol{\theta})\boldsymbol{x} \tag{46}$$

subject to

$$\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{\theta}) \ : \ \boldsymbol{\lambda} \tag{47}$$

then, using the chain rule and (38), the sensitivities become

$$\begin{aligned}
\frac{\partial Z}{\partial \theta_r} &= \sum_j x_j \frac{\partial c_j(\boldsymbol{\theta})}{\partial \theta_r} + \sum_{i,j} \lambda_i x_j \frac{\partial a_{ij}(\boldsymbol{\theta})}{\partial \theta_r} - \sum_i \lambda_i \frac{\partial b_i(\boldsymbol{\theta})}{\partial \theta_r} \\
\frac{\partial x_j}{\partial \theta_r} &= -\sum_{i,k} a^{ji} x_k \frac{\partial a_{ik}(\boldsymbol{\theta})}{\partial \theta_r} + \sum_i a^{ji} \frac{\partial b_i(\boldsymbol{\theta})}{\partial \theta_r} \\
\frac{\partial \lambda_i}{\partial \theta_r} &= -\sum_j a^{ji} \frac{\partial c_j(\boldsymbol{\theta})}{\partial \theta_r} - \sum_{\ell,j} a^{ji} \lambda_\ell \frac{\partial a_{\ell j}(\boldsymbol{\theta})}{\partial \theta_r}
\end{aligned} \tag{48}$$

or in matrix form

$$
\begin{aligned}
\frac{\partial Z}{\partial \boldsymbol{\theta}} &= \boldsymbol{x}^T \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\theta}} - \boldsymbol{\lambda}^T \left(\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\theta}} - \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\theta}} \boldsymbol{x} \right) \\
\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} &= \boldsymbol{A}^{-1} \left(\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\theta}} - \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\theta}} \boldsymbol{x} \right) \\
\frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} &= -\left(\boldsymbol{A}^{-1} \right)^T \left(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\theta}} + \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\theta}} \right)^T \boldsymbol{\lambda} \right).
\end{aligned} \tag{49}
$$

Note that this is the general case and that the sensitivities in (38) are particular cases of it.

3.2.1 The Goldstein and Yudin Example

In this section we study an example in which the data, including cost coefficients and constraints, depend on a single parameter θ. The problem will be solved in symbolic form, i.e., not for a fixed value of θ, but in terms of θ.

Consider the following example given in Goldstein and Yudin [19]):

$$
\underset{x_1, x_2, x_3, x_4}{\text{Minimize}} \quad Z(\theta) = (30 + 6\theta)x_1 + (50 + 7\theta)x_2
$$

subject to

$$
\begin{aligned}
(14 + 2\theta)x_1 + (4 + \theta)x_2 + x_3 - 14 - \theta &= 0 \ : \ \lambda_1 & (50) \\
(150 + 3\theta)x_1 + (200 + 4\theta)x_2 - x_4 - 200 - 2\theta &= 0 \ : \ \lambda_2 & (51) \\
x_1, x_2, x_3, x_4 &\geq 0 \ : \ \boldsymbol{\mu}, & (52)
\end{aligned}
$$

whose solution for $-3 \leq \theta \leq 10$, according to Goldstein and Yudin [19], is

$$
\boldsymbol{x}^*(\theta) = \begin{pmatrix} \dfrac{2\ (1000 + 24\,\theta + \theta^2)}{2200 + 294\,\theta + 5\,\theta^2} & \dfrac{700 + 236\,\theta + \theta^2}{2200 + 294\,\theta + 5\,\theta^2} & 0 & 0 \end{pmatrix}^T, \tag{53}
$$

where $\boldsymbol{x}^*(\theta) = (\, x_1^*(\theta) \quad x_2^*(\theta) \quad x_3^*(\theta) \quad x_4^*(\theta)\,)^T$, and the dual variables $\boldsymbol{\lambda}^*(\theta)$ become

$$
-\begin{pmatrix} \dfrac{3\theta - 30}{44 + 5\,\theta} & \dfrac{580 + 144\theta + 8\theta^2}{2200 + 294\,\theta + 5\,\theta^2} & \dfrac{3\theta - 30}{44 + 5\,\theta} & \dfrac{-580 - 144\theta - 8\theta^2}{2200 + 294\,\theta + 5\,\theta^2} \end{pmatrix}^T, \tag{54}
$$

where $\boldsymbol{\lambda}^*(\theta) = (\, \lambda_1^*(\theta) \quad \lambda_2^*(\theta) \quad \lambda_3^*(\theta) \quad \lambda_4^*(\theta)\,)^T$ and λ_3^* and λ_4^* refer to the dual variables associated with the active inequality constraints μ_3^* and μ_4^*, respectively. Note that $\mu_1^*(\theta) = \mu_2^*(\theta) = 0$. Finally, the objective function optimum value is

$$
Z^*(\theta) = \frac{95000 + 30140\,\theta + 2050\,\theta^2 + 19\,\theta^3}{2200 + 294\,\theta + 5\,\theta^2}. \tag{55}
$$

Considering the matrices

$$
\frac{\partial \boldsymbol{a}}{\partial \theta} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \frac{\partial \boldsymbol{b}}{\partial \theta} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}; \quad \frac{\partial \boldsymbol{c}}{\partial \theta} = \begin{bmatrix} 6 \\ 7 \\ 0 \\ 0 \end{bmatrix},
$$

and using formulas (48) and (49) one gets the following sensitivities:

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial \theta} \\ \frac{\partial x_2^*}{\partial \theta} \\ \frac{\partial x_3^*}{\partial \theta} \\ \frac{\partial x_4^*}{\partial \theta} \\ \frac{\partial \lambda_1^*}{\partial \theta} \\ \frac{\partial \lambda_2^*}{\partial \theta} \\ \frac{\partial \lambda_3^*}{\partial \theta} \\ \frac{\partial \lambda_4^*}{\partial \theta} \\ \frac{\partial Z^*}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{4\,(-120600 - 2800\,\theta + 87\,\theta^2)}{(50+\theta)^2\,(44+5\,\theta)^2} \\ \frac{-2\,(-156700 + 1300\,\theta + 443\,\theta^2)}{(50+\theta)^2\,(44+5\,\theta)^2} \\ 0 \\ 0 \\ -\frac{282}{(44+5\,\theta)^2} \\ -\frac{24\,(6095 + 1225\,\theta + 68\,\theta^2)}{(50+\theta)^2\,(44+5\,\theta)^2} \\ -\frac{282}{(44+5\,\theta)^2} \\ \frac{24\,(6095 + 1225\,\theta + 68\,\theta^2)}{(50+\theta)^2\,(44+5\,\theta)^2} \\ \frac{38378000 + 8070000\,\theta + 577400\,\theta^2 + 11172\,\theta^3 + 95\,\theta^4}{(50+\theta)^2\,(44+5\,\theta)^2} \end{bmatrix},$$

which are valid in the range $\theta \in (-3, 10)$ and coincide with those obtained from taking partial derivatives in (53), (54) and (55).

For the particular case $\theta = 0$, they become:

$$\boldsymbol{s}^{*T} = \begin{bmatrix} -\frac{603}{6050} & \frac{1567}{24200} & 0 & 0 & -\frac{141}{968} & -\frac{3657}{121000} & -\frac{141}{968} & \frac{3657}{121000} & \frac{19189}{2420} \end{bmatrix}, \tag{56}$$

where

$$\boldsymbol{s}^{*T} = \begin{bmatrix} \frac{\partial x_1^*}{\partial \theta} & \frac{\partial x_2^*}{\partial \theta} & \frac{\partial x_3^*}{\partial \theta} & \frac{\partial x_4^*}{\partial \theta} & \frac{\partial \lambda_1^*}{\partial \theta} & \frac{\partial \lambda_2^*}{\partial \theta} & \frac{\partial \lambda_3^*}{\partial \theta} & \frac{\partial \lambda_4^*}{\partial \theta} & \frac{\partial Z^*}{\partial \theta} \end{bmatrix}. \tag{57}$$

Note that the matrix form of (49) allows one an easy treatment of this parametric sensitivity problem even in symbolic form.

4 Application to Nonlinear Programming

In this section we extend the results of the previous section to the case of nonlinear programming. Consider the nonlinear programming problem

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad Z = f(\boldsymbol{x};\boldsymbol{\theta}); \tag{58}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x};\boldsymbol{\theta}) = \boldsymbol{0}, \tag{59}$$

where $\boldsymbol{\theta}$ is a parameter vector, and denote $\boldsymbol{\lambda}$ the dual variables. Assume that the problem is regular for $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, and that all the constraints are active. More precisely, assume that the nonlinear problem has been solved and that one knows its optimal solution $\boldsymbol{x}^*$ and its dual solution

$\boldsymbol{\lambda}^*$, and that one has removed the inactive constraints and all active inequality constraints have been converted to equality constraints, so that (59) contains all equality constraints and the active inequality constraints of the original problem converted to equality constraints. We also assume that all dual variables coming from inequalities are positive.

In the first case indicated in Section 2, expressions (25) into (29) are valid replacing $\boldsymbol{Q}$ by $\boldsymbol{H}$, and then the sensitivities are given by (27) to (29)

In the second case indicated in Section 2, if the matrix $\boldsymbol{F_{xx}}$ is invertible and $\boldsymbol{H_x}$ is full row rank, then $\boldsymbol{B} = -\boldsymbol{H_x}\boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T$ is also invertible and we have

$$\boldsymbol{U}^{-1} = \left[\begin{array}{c|c|c} \boldsymbol{0}_{n\times 1} & \left(\boldsymbol{I} + \boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_x}\right)\boldsymbol{F_{xx}}^{-1} & -\boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1} \\ \hline \boldsymbol{0}_{\ell\times 1} & -\boldsymbol{B}^{-1}\boldsymbol{H_x}\boldsymbol{F_{xx}}^{-1} & \boldsymbol{B}^{-1} \\ \hline -1_{1\times 1} & \boldsymbol{F_x}\left(\boldsymbol{I} + \boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_x}\right)\boldsymbol{F_{xx}}^{-1} & -\boldsymbol{F_x}\boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1} \end{array}\right], \quad (60)$$

from which we get the alternative closed formulas

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} = -\left(\boldsymbol{I} + \boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_x}\right)\boldsymbol{F_{xx}}^{-1}\boldsymbol{F_{x\theta}} + \boldsymbol{F_{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_\theta} \quad (61)$$

$$\frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} = \boldsymbol{B}^{-1}\boldsymbol{H_x}\boldsymbol{F_{xx}}^{-1}\boldsymbol{F_{x\theta}} - \boldsymbol{B}^{-1}\boldsymbol{H_\theta} \quad (62)$$

$$\frac{\partial Z}{\partial \boldsymbol{\theta}} = \boldsymbol{F_\theta} - \boldsymbol{F_x}\left(\boldsymbol{I} + \boldsymbol{F}_{\boldsymbol{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_x}\right)\boldsymbol{F_{xx}}^{-1}\boldsymbol{F_{x\theta}} + \boldsymbol{F_x}\boldsymbol{F}_{\boldsymbol{xx}}^{-1}\boldsymbol{H}_{\boldsymbol{x}}^T\boldsymbol{B}^{-1}\boldsymbol{H_\theta}. \quad (63)$$

4.1 A non-linear example

For the sake of illustration, consider the following simple non-linear programming problem:

$$\underset{x_1, x_2}{\text{Minimize}} \quad Z = -\theta^2 x_1 + \theta x_2^2 \quad (64)$$

subject to

$$\theta^3(x_1+1)^2 + (x_2+1)^2 - 2 \;\leq\; 0 \quad (65)$$

$$-\theta^2(x_1-1)^2 - (2\theta-1)(x_2+1)^2 + 2 \;\leq\; 0. \quad (66)$$

This problem is used to illustrate the two cases of Section 2.

To illustrate the first case, assume that $\theta = 1$. Then, its optimal solution is $x_1^* = x_2^* = 0$ and the corresponding dual solution is $\eta_1^* = \eta_2^* = 1/4$.

The required matrices to perform the sensitivity analysis are:

$$\begin{array}{lcll} \boldsymbol{F_x} & = & \begin{pmatrix} -\theta^2 & 2\,\theta\, x_2 \end{pmatrix} & \Rightarrow \boldsymbol{F}_{\boldsymbol{x}}^* = \begin{pmatrix} -1 & 0 \end{pmatrix} \\ \boldsymbol{F_{xx}} & = & \begin{pmatrix} \frac{(-1+\theta)\,\theta^2}{2} & 0 \\ 0 & 1+\theta \end{pmatrix} & \Rightarrow \boldsymbol{F}_{\boldsymbol{xx}}^* = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \\ \boldsymbol{H_x} & = & \begin{pmatrix} 2\,\theta^3\,(1+x_1) & 2\,(1+x_2) \\ -2\,\theta^2\,(-1+x_1) & -2\,(-1+2\,\theta)\,(1+x_2) \end{pmatrix} & \Rightarrow \boldsymbol{H}_{\boldsymbol{x}}^* = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \\ \boldsymbol{F_\theta} & = & \begin{pmatrix} -2\,\theta\, x_1 + x_2^2 \end{pmatrix} & \Rightarrow \boldsymbol{F}_{\boldsymbol{\theta}}^* = \begin{pmatrix} 0 \end{pmatrix} \\ \boldsymbol{F_{x\theta}} & = & \begin{pmatrix} \frac{\theta\,(-2+3\,\theta)\,(1+x_1)}{2} \\ -1+x_2 \end{pmatrix} & \Rightarrow \boldsymbol{F}_{\boldsymbol{x\theta}}^* = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \\ \boldsymbol{H_\theta} & = & \begin{pmatrix} 3\,\theta^2\,(1+x_1)^2 \\ -2\,\theta\,(-1+x_1)^2 - 2\,(1+x_2)^2 \end{pmatrix} & \Rightarrow \boldsymbol{H}_{\boldsymbol{\theta}}^* = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \end{array}$$

where the asterisks refer to the matrices resulting after replacing the optimal values $x_1^* = x_2^* = 0$ and $\theta = 1$.

Since $\boldsymbol{H_x}$ is invertible, we can apply equations (27)-(29) to obtain:

$$\begin{bmatrix} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} \\ -- \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} \\ -- \\ \dfrac{\partial Z}{\partial \boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} 1/4 \\ -7/4 \\ --- \\ 1 \\ -5/4 \\ --- \\ -1/4 \end{bmatrix},$$

which are the sensitivities sought after.

To illustrate the second case, assume that $\theta = 2$. Then, its optimal solution is $x_1^* = -0.58069108$ and $x_2^* = -0.22964926$ and the corresponding dual solution is $\mu_1^* = 0.59621914$ and $\mu_2^* = 0$, with an optimal value $z = 2.42824$. In this case the second constraint is not active, and (61), (62) and (63) lead to the following sensitivities

$$\begin{bmatrix} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} \\ -- \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\theta}} \\ -- \\ \dfrac{\partial Z}{\partial \boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} -0.323741 \\ 0.040321 \\ --- \\ 0.162221 \\ 0 \\ --- \\ 3.63343 \end{bmatrix}.$$

5 Conclusions

In the context of sensitivity analysis in mathematical programming, the main conclusions derived from the work reported in this paper are the following:

1. A set of closed formulas (see (38)) has been given for obtaining the sensitivities of the objective function optimal value, and the primal and dual variables of a LPP with respect to the cost coefficients c_j, the constraint coefficients a_{ij} and the right hand side coefficients b_i, when the number of non-redundant active constraints coincides with the number of variables.

2. A set of closed formulas (see (48) and (49)) for obtaining the sensitivities of the objective function optimal value, and the primal and dual variables of a LPP with respect to a given parameter θ_r that can appear exclusively or simultaneously in the objective function and in the constraints, has been given when the number of non-redundant active constraints coincides with the number of variables.

3. Closed formulas for obtaining the corresponding sensitivities in a general NLPP have been given for the following two cases: (a) when the number of active constraints (equalities plus inequalities) coincides with the number of variables, and the matrix $\boldsymbol{Q}$ is invertible (see (27)-(29)), and (b) when $\boldsymbol{F_{xx}}$ is positive definite and $\boldsymbol{Q}$ is full row rank (see (61)-(63)).

4. All the above formulas are derived in a compact form and simplify the process of performing a sensitivity analysis in NLPPs, as it has been shown in the examples.

5. For large size problems, the usefulness of the proposed formulas for calculating the sensitivities, is subject to the possibility of inverting the corresponding matrices or solving the associated systems of equations. Even though this can be a difficult numerical problem, note that normally matrix $\boldsymbol{U}$ is highly sparse as a result of the sparsity of its building blocks so that it is easily factorized using sparse-oriented LU algorithms, which alleviate the problem.

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