A Sensitivity Analysis Method to Compute the Residual Covariance Matrix

E. Caro[a], A.J. Conejo[a,*], R. Mínguez[b]

[a] *Department of Electrical Engineering, Univ. Castilla-La Mancha, Campus Universitario s/n, 13071 Ciudad Real, Spain*
[b] *Environmental Hydraulics Institute, Univ. of Cantabria, Cantabria, Spain*

Abstract

In state estimation, the covariance matrix of residuals is used to compute the normalized residuals and to detect erroneous measurements. This paper describes an estimator-independent method based on sensitivity analysis that allows computing the residual covariance matrix. This method is suitable for most solution approaches based on mathematical programming procedures. Several case studies illustrate the technique proposed. Relevant conclusions are finally drawn.
Keywords: Power system state estimation, Residual covariance matrix, Sensitivity analysis

1. Introduction

1.1. Motivation and Aim

State estimation consists in processing a given set of measurements to obtain the optimal estimate of the power system state. Several state estimation methods are proposed in the technical literature. Most of them are based on solving an optimization problem, such as the following methods: the Weighted Least Squares (WLS), the Least Absolute Value (LAV), the Least Median of

*Corresponding author. Phone: +34 926 295433; Fax: +34 926 295361.
Email addresses: Eduardo.Caro@uclm.es (E. Caro), Antonio.Conejo@uclm.es (A.J. Conejo), Roberto.Minguez@unican.es (R. Mínguez)

Squares (LMS), the Least Trimmed of Squares (LTS), the Quadratic-Constant Criterion (QCC), and the Quadratic-Linear Criterion (QLC).

Measurements may contain gross errors due to various reasons. Thus, an essential feature of any state estimator is to detect those gross measurement errors, and, if possible, to identify and eliminate them. In general, bad measurement identification procedures rely on the residual covariance matrix and on the subsequent residual normalization. However, residual covariance matrix computation techniques differ across the methods. Moreover, these techniques usually compute an approximate residual covariance matrix using a first-order approximation and generally disregarding constraints. To overcome these drawbacks is the aim of this paper; i.e., to propose a novel estimator-independent procedure to compute accurately the residual covariance matrix.

1.2. Literature Review and Contribution

The technical literature is rich in references pertaining to state estimation techniques and algorithms [1]–[20]. Particularly notorious is the Weighted Least Squares estimator, which is a non-robust method exhaustively studied in the literature [1]–[12].

Alternative approaches rely on the use of robust estimators, i.e., procedures less sensitive to bad measurements or outliers than the WLS technique. Some of them are based on minimizing a non-quadratic function of measurement residuals. The Quadratic-Constant and Quadratic-Linear Criterions belong to this category [14, 13]. The Least Absolute Value method also belongs to this category and has gained widespread relevance thanks to its implicit bad data rejection property [15]–[17].

The Least Median of Squares [18]–[19] and the Least Trimmed of Squares [20] estimators are members of the family known as high-breakdown point estimators. These methods are also capable of eliminating the effect of leverage points (measurements that critically affect the estimator, [11]).

For any particular estimator, the technical literature provides estimator-specific techniques to compute the residual covariance matrix. For example,

[21], [22], and [15] report methods to compute this matrix for the unconstrained-WLS, constrained-WLS, and LAV estimators, respectively. For case of high-breakdown point estimators, [23] and [24] provide procedures to compute the asymptotic covariance residual matrices. To the best of our knowledge, no previous work proposes an estimator-independent method to compute this matrix.

The specific contribution of this paper is to provide an estimator-independent method to compute the residual covariance matrix for most optimization-based state estimators, which considers constraints and second-order derivatives.

1.3. Paper Organization

The rest of this paper is organized as follows. In Section 2, the state estimation problem is formulated as a general mathematical programming problem. In Section 3, the methodology for obtaining the residual covariance matrix and the normalized residuals is developed. Section 4 provides the expressions for calculating the derivatives of the estimates of the state variables with respect to the measurement values, which are needed for the estimation of the residual covariance matrix. In Section 5, the proposed technique is particularized for the well-known WLS and LAV methods. Section 6 provides results from three case studies to illustrate the performance of the proposed method. Finally, Section 7 provides some relevant conclusions.

2. State Estimation Formulation

Most state estimation models in practical use are stated as mathematical programming problems. These problems are formulated, in general, as:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad J(\boldsymbol{y}) \tag{1}$$

subject to:

$$\boldsymbol{l}(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}) \leq \boldsymbol{0} \tag{3}$$

where $\boldsymbol{z}$ is the $m \times 1$ measurement vector, $\boldsymbol{x}$ is the $n \times 1$ state-variable vector (variables to be estimated), $\boldsymbol{y}$ is the difference vector between the measurement and the functional vectors, i.e., $\boldsymbol{y} = \boldsymbol{z} - \boldsymbol{h}(\boldsymbol{x})$, $J(\boldsymbol{y})$ is the objective function defined by the estimator, $\boldsymbol{l}(\boldsymbol{x}, \boldsymbol{z})$ are equality constraints, e.g., to model zero-injection buses, and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$ are inequality constraints, e.g., physical limits or constraints for transforming the LAV model into an equivalent one eliminating the absolute value function (see Section 5.3). Parameters p and q correspond to the number of equality and inequality constraints, respectively.

The solution of problem (1)–(3) provides the optimal estimate of the system state, $\hat{\boldsymbol{x}}$, which is assumed to be close enough to the true state $\boldsymbol{x}^{\text{true}}$. The residual vector $\boldsymbol{r}$ is defined as:

$$\boldsymbol{r} = \boldsymbol{z} - \boldsymbol{h}(\hat{\boldsymbol{x}}) \ . \tag{4}$$

Note that $\boldsymbol{r} = \boldsymbol{y}|_{\boldsymbol{x}=\hat{\boldsymbol{x}}}$.

Using the general model (1)–(3) this paper provides a procedure to compute the residual covariance matrix based on sensitivity analysis [25].

3. Residual Covariance Matrix and Residual Normalization

Using a first-order Taylor expansion of function $\boldsymbol{h}(\boldsymbol{x})$ around the optimal state vector $\hat{\boldsymbol{x}}$, the differential residual vector is obtained from (4) as:

$$d\boldsymbol{r} \;=\; d\boldsymbol{z} - \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}} d\hat{\boldsymbol{x}} \;=\; d\boldsymbol{z} - \boldsymbol{H} d\hat{\boldsymbol{x}} \tag{5}$$

where $\boldsymbol{H}$ is the $m \times n$ Jacobian measurement matrix evaluated at $\hat{\boldsymbol{x}}$.

From (5), it readily follows:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}} \;=\; \boldsymbol{I} - \boldsymbol{H}\left.\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}} \;=\; \boldsymbol{I} - \boldsymbol{H}\boldsymbol{M}_{xz} \;=\; \boldsymbol{S} \tag{6}$$

where matrix $\boldsymbol{M}_{xz}$ is made of the derivatives of the state estimator vector $\boldsymbol{x}$ with respect to measurements $\boldsymbol{z}$ evaluated at $\hat{\boldsymbol{x}}$, matrix $\boldsymbol{I}$ is the m-dimensional identity matrix, and matrix $\boldsymbol{S}$ is known as the residual *sensitivity* matrix.

Note that (6) allows calculating matrix $\boldsymbol{S}$ for the general model (1)–(3).

The linear transformation from $\boldsymbol{z}$ to $\boldsymbol{r}$ at the optimum is obtained throughout the integration of (6):

$$\boldsymbol{r} \quad = \quad \boldsymbol{S}\boldsymbol{z} + \boldsymbol{k} \tag{7}$$

where $\boldsymbol{k}$ is the integration constant vector.

The expected value of the residual vector in (7) is:

$$\mathrm{E}[\boldsymbol{r}] \quad = \quad \boldsymbol{S}\mathrm{E}[\boldsymbol{z}] + \boldsymbol{k} \tag{8}$$

and, subtracting (8) from (7), it readily follows:

$$\boldsymbol{r} - \mathrm{E}[\boldsymbol{r}] \quad = \quad \boldsymbol{S}\big(\boldsymbol{z} - \mathrm{E}\,[\boldsymbol{z}]\big)\ . \tag{9}$$

From (9), the residual covariance matrix $\boldsymbol{\Omega}$ is:

$$\begin{aligned}
\boldsymbol{\Omega} \quad &= \quad \mathrm{E}\left[\big(\boldsymbol{r} - \mathrm{E}[\boldsymbol{r}]\big)\big(\boldsymbol{r} - \mathrm{E}[\boldsymbol{r}]\big)^T\right] \\
&= \quad \mathrm{E}\left[\big(\boldsymbol{S}\big(\boldsymbol{z} - \mathrm{E}[\boldsymbol{z}]\big)\big)\,\big(\boldsymbol{S}\big(\boldsymbol{z} - \mathrm{E}[\boldsymbol{z}]\big)\big)^T\right] \\
&= \quad \boldsymbol{S}\mathrm{E}\left[\big(\boldsymbol{z} - \mathrm{E}[\boldsymbol{z}]\big)\big(\boldsymbol{z} - \mathrm{E}[\boldsymbol{z}]\big)^T\right]\boldsymbol{S}^T \\
&= \quad \boldsymbol{S}\boldsymbol{C}_z\boldsymbol{S}^T
\end{aligned} \tag{10}$$

where matrix $\boldsymbol{C}_z$ is the measurement covariance matrix.

Therefore, considering (6), the general expression of matrix $\boldsymbol{\Omega}$ is:

$$\boldsymbol{\Omega} \quad = \quad (\boldsymbol{I} - \boldsymbol{H}\boldsymbol{M}_{xz})\boldsymbol{C}_z(\boldsymbol{I} - \boldsymbol{H}\boldsymbol{M}_{xz})^T\ . \tag{11}$$

Note that matrix $\boldsymbol{M}_{xz}$ depends on the estimator used, whereas matrix $\boldsymbol{H}$ is computed in the same way for all estimators. The main contribution of this paper is to provide an estimator-independent procedure to obtain matrix $\boldsymbol{M}_{xz}$.

Finally, from (4) and (11), normalized residuals are computed as

$$r_i^N = \frac{|r_i|}{\sqrt{[\boldsymbol{\Omega}]_{(i,i)}}} = \frac{|z_i - h_i(\hat{\boldsymbol{x}})|}{\sqrt{[\boldsymbol{\Omega}]_{(i,i)}}} \quad i = 1, \ldots, m\ . \tag{12}$$

Vector $\boldsymbol{r}^N$ corresponds to the normalized residuals, and it can be used straightforwardly for bad data identification.

4. Derivatives of the State Variables with Respect to the Measurements

As shown in Section 3, to compute the sensitivity matrix $\boldsymbol{S}$, which allows calculating the residual covariance matrix $\boldsymbol{\Omega}$, matrix $\boldsymbol{M}_{xz}$ is required. This matrix is obtained below, based on sensitivity analysis results reported in [26]. The technique to obtain matrix $\boldsymbol{M}_{xz}$ constitutes the main contribution of this paper.

The optimal primal/dual solution of problem (1)–(3) is denoted as $(\hat{\boldsymbol{x}}, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is the dual variable vector related to both equality and active inequality constraints. The Karush-Kuhn-Tucker (KKT) first order optimality conditions for problem (1)–(3) are:

$$\nabla_{\boldsymbol{x}} J(\hat{\boldsymbol{x}}, \boldsymbol{z}) + \boldsymbol{\lambda}^T \, \nabla_{\boldsymbol{x}} \boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{z}) = \boldsymbol{0} \quad (13)$$

$$\boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{z}) = \boldsymbol{0} \quad (14)$$

where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z})$ corresponds to both equality and active inequality constraints.

Note that equality and active inequality constraints are treated similarly, and inactive inequality constraints are disregarded, which is possible once the optimal solution is known. This assumption implies that the same constraints remain active after any infinitesimal data variation, which is an appropriate assumption knowing that the analysis is just local and assuming that the derivatives of the objective function are continuous at the optimum.

Considering the aforementioned requirements, the proposed technique can be applied to the Weighted Least Squares, Least Absolute Value, Quadratic-Constant and Quadratic-Linear Criterion, among others. However, the objective functions of Least Median of Squares or Least Trimmed of Squares estimators are not continuous and, therefore, the proposed approach cannot be applied to these particular estimators.

To obtain the required derivatives, we perturb or modify $\hat{\boldsymbol{x}}$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$, in such a way that the KKT conditions hold, [26]. To this end, we differentiate the optimality conditions (13)–(14), obtaining the following linear system of equations:

$$\left[\begin{array}{c|c|c} \boldsymbol{J}_{xx} & \boldsymbol{J}_{xz} & \boldsymbol{F}_x^T \\ \hline \boldsymbol{F}_x & \boldsymbol{F}_z & \boldsymbol{0} \end{array}\right] \begin{bmatrix} d\boldsymbol{x} \\ d\boldsymbol{z} \\ d\boldsymbol{\lambda} \end{bmatrix} = \boldsymbol{0} \tag{15}$$

where the vectors and submatrices in (15) are evaluated at the optimal solution, $(\hat{\boldsymbol{x}}, \boldsymbol{\lambda})$. The required matrices in (15) are:

$$\begin{aligned} \boldsymbol{J}_{xx(n\times n)} &= \nabla_{\boldsymbol{xx}} J(\hat{\boldsymbol{x}}, \boldsymbol{z}) \\ &+ \sum_{k=1}^{p+q_\Gamma} \lambda_k \nabla_{\boldsymbol{xx}} f_k(\hat{\boldsymbol{x}}, \boldsymbol{z}) \qquad (16) \\ \boldsymbol{J}_{xz(n\times m)} &= \nabla_{\boldsymbol{xz}} J(\hat{\boldsymbol{x}}, \boldsymbol{z}) \\ &+ \sum_{k=1}^{p+q_\Gamma} \lambda_k \nabla_{\boldsymbol{xz}} f_k(\hat{\boldsymbol{x}}, \boldsymbol{z}) \qquad (17) \\ \boldsymbol{F}_{x((p+q_\Gamma)\times n)} &= [\nabla_{\boldsymbol{x}} \boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{z})]^T \qquad (18) \\ \boldsymbol{F}_{z((p+q_\Gamma)\times m)} &= [\nabla_{\boldsymbol{z}} \boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{z})]^T \qquad (19) \end{aligned}$$

where p and q_Γ are the number of equality and active inequality constraints, respectively. Matrix dimensions are indicated in parenthesis.

To compute derivatives with respect to the components of the measurement vector $\boldsymbol{z}$, system (15) can be rewritten as

$$\boldsymbol{U} \begin{bmatrix} d\boldsymbol{x} \\ d\boldsymbol{\lambda} \end{bmatrix} = \boldsymbol{S}_z \, d\boldsymbol{z} \tag{20}$$

where

$$\boldsymbol{U} = \left[\begin{array}{c|c} \boldsymbol{J}_{xx} & \boldsymbol{F}_x^T \\ \hline \boldsymbol{F}_x & \boldsymbol{0} \end{array}\right] \tag{21}$$

and

$$\boldsymbol{S}_z = -\begin{bmatrix} \boldsymbol{J}_{xz} \\ \boldsymbol{F}_z \end{bmatrix} \tag{22}$$

and therefore

$$\begin{bmatrix} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}} \\ \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{z}} \end{bmatrix} = \boldsymbol{U}^{-1}\boldsymbol{S}_z \ . \tag{23}$$

Finally, $\boldsymbol{M}_{xz} = \left.\dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}\right|_{\hat{\boldsymbol{x}}}$ is the block matrix from (23) corresponding to the required partial derivatives. Matrix $\boldsymbol{M}_{xz}$ includes information of second-order derivatives, through equations (16) and (17), as well as the effect of equality and active inequality constraints. Additional computational details and simplifications of expression (23) under certain conditions are provided in [26]–[27].

Note that the main contribution of this paper is the estimator-independent computation of matrix $\boldsymbol{M}_{xz}$ through (23). To the best of our knowledge, no prior work proposes such unified and accurate method. Note also that (11) and (23) are general expressions that can be straightforwardly used with most state estimators.

From the computational point of view, matrices $\boldsymbol{U}$ and $\boldsymbol{S}_z$ are highly sparse as a result of the network connectivity, leading to a low percentage of non-zero elements [25]. Note that matrix $\boldsymbol{U}$ can be straightforwardly factorized using sparse-oriented LU algorithms exploding its symmetric properties, and the linear system (23) can be efficiently solved using forward and backward substitution.

Regarding the required information on derivatives, most of them result from the estimation problem, and the remaining derivatives, mostly associated with measurements, can be easily computed either analytically or numerically.

5. Particular Cases

In this section, the general expressions derived in Section 4 are particularized for the two most common state estimators: WLS and LAV, and the resulting expressions are then compared with those available in the technical literature.

5.1. Unconstrained WLS Case

If the objective function in problem (1)–(3) corresponds to the WLS formulation,

$$J(\boldsymbol{y}) = \frac{1}{2}\big[\boldsymbol{z} - \boldsymbol{h}(\boldsymbol{x})\big]^T \boldsymbol{C}_z^{-1}\big[\boldsymbol{z} - \boldsymbol{h}(\boldsymbol{x})\big] \tag{24}$$

and neither $\boldsymbol{l}(\boldsymbol{x},\boldsymbol{z})$ nor $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{z})$ constraints are taken into account, matrices in (15) become:

$$\boldsymbol{J}_{xx} = \nabla_{\boldsymbol{xx}} J(\hat{\boldsymbol{x}},\boldsymbol{z}) \tag{25}$$

$$\boldsymbol{J}_{xz} = -\boldsymbol{H}^T\boldsymbol{C}_z^{-1} \tag{26}$$

$$\boldsymbol{F}_x = \emptyset \tag{27}$$

$$\boldsymbol{F}_z = \emptyset\,. \tag{28}$$

Using (21)–(23), we obtain the final expression:

$$\begin{aligned}\boldsymbol{M}_{xz} &= \boldsymbol{U}^{-1}\boldsymbol{S}_z \;=\; [\boldsymbol{J}_{xx}]^{-1}[-\boldsymbol{J}_{xz}] \\ &= (\boldsymbol{J}_{xx})^{-1}\boldsymbol{H}^T\boldsymbol{C}_z^{-1}\end{aligned} \tag{29}$$

and matrix $\boldsymbol{\Omega}$ can be straightforwardly computed using (11) and (29). This final expression is novel and dissimilar from others proposed in the technical literature [21], since it considers second order derivatives.

5.1.1. Comparison with the Approach in [22]

The traditional approach to compute matrix $\boldsymbol{\Omega}$ as stated in [21] is an approximation, which can be derived from (29) assuming that the second-order derivatives of the objective function are null, i.e., $J(\boldsymbol{x},\boldsymbol{z})$ is assumed to be a multi-linear function. This approximation for element (j,k) of $\boldsymbol{J}_{xx}$ is detailed below:

$$\begin{aligned}[\boldsymbol{J}_{xx}]_{(j,k)} &= \left.\frac{\partial^2 J(\boldsymbol{x},\boldsymbol{z})}{\partial x_j \partial x_k}\right|_{\hat{\boldsymbol{x}}} \\ &= \sum_{i=1}^{m} W_{ii}\left.\left[\frac{\partial h_i(\boldsymbol{x})}{\partial x_j}\frac{\partial h_i(\boldsymbol{x})}{\partial x_k} + \frac{\partial^2 h_i(\boldsymbol{x})}{\partial x_j \partial x_k}\left(h_i(\boldsymbol{x}) - z_i\right)\right]\right|_{\hat{\boldsymbol{x}}} \\ &\approx \sum_{i=1}^{m} W_{ii}\left.\left[\frac{\partial h_i(\boldsymbol{x})}{\partial x_j}\frac{\partial h_i(\boldsymbol{x})}{\partial x_k}\right]\right|_{\hat{\boldsymbol{x}}} = \left[\boldsymbol{H}^T\boldsymbol{C}_z^{-1}\boldsymbol{H}\right]_{(j,k)}\end{aligned} \tag{30}$$

where $\boldsymbol{W}$ is the weighting matrix, generally computed as $\boldsymbol{W} = \boldsymbol{C}_z^{-1}$, [11].

From (30), note that matrix $\boldsymbol{C}_z$ is implicitly assumed to be diagonal, i.e., measurements are considered independent, which is a common assumption in state estimation. If measurement dependencies are considered [28, 29] (i.e., matrix $\boldsymbol{C}_z$ is non-diagonal), approximation (30) still holds true.

It is shown below that the traditional approach to compute matrix $\boldsymbol{\Omega}$ is an approximation of the method proposed in this paper. Using approximate expression (30) in equations (6), (10), and (29), it readily follows:

$$\boldsymbol{M}_{xz} = (\boldsymbol{H}^T\boldsymbol{C}_z^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{C}_z^{-1} \tag{31}$$

$$\boldsymbol{S} = \boldsymbol{I} - \boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{C}_z^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{C}_z^{-1} \tag{32}$$

$$\boldsymbol{\Omega} = \boldsymbol{S}\boldsymbol{C}_z\boldsymbol{S}^T = \boldsymbol{S}\boldsymbol{C}_z \tag{33}$$

which corresponds to the expression traditionally used in the technical literature, [21]. It can be shown that matrix $\boldsymbol{S}$ is an idempotent matrix [11], property used to derive the final expression in (33).

Note also that in case of considering equality constraints or active inequality constraints, the proposed technique to compute matrix $\boldsymbol{\Omega}$, based on (23), differs from (33).

5.2. Equality-Constrained WLS Case

The technical literature also provides a method to compute the residual sensitivity matrix $\boldsymbol{S}$ including equality and active inequality constraints in the WLS estimator, [22]. As shown below, this approach is also an approximation of the one presented in this paper.

If equality and active inequality constraints are included in the analysis, equations (24)–(26) still hold, and:

$$\boldsymbol{F}_x = \boldsymbol{F} \quad , \quad \boldsymbol{F}_z = \emptyset \tag{34}$$

where $\boldsymbol{F}$ is the Jacobian matrix of equations $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z})$ with respect to the state variable vector evaluated at the optimal estimate $\hat{\boldsymbol{x}}$, i.e.,

$$\boldsymbol{F} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}} . \tag{35}$$

Therefore, using (23), $\boldsymbol{M}_{xz} = \boldsymbol{E}_1 \boldsymbol{H}^T \boldsymbol{C}_z^{-1}$, where $\boldsymbol{E}_1$ corresponds to the upper-left quadrant of matrix $\boldsymbol{U}^{-1}$, i.e.,

$$\boldsymbol{U}^{-1} = \left[\begin{array}{c|c} \boldsymbol{J}_{xx} & \boldsymbol{F}^T \\ \hline \boldsymbol{F} & \emptyset \end{array}\right]^{-1} = \left[\begin{array}{c|c} \boldsymbol{E}_1 & \boldsymbol{E}_2^T \\ \hline \boldsymbol{E}_2 & \boldsymbol{E}_3 \end{array}\right] . \tag{36}$$

Finally, the expression to compute the residual sensitivity matrix through the proposed method becomes

$$\boldsymbol{S} = \boldsymbol{I} - \boldsymbol{H}\boldsymbol{E}_1 \boldsymbol{H}^T \boldsymbol{C}_z^{-1} . \tag{37}$$

5.2.1. Comparison with the Approach in [22]

The method used to compute matrix $\boldsymbol{\Omega}$ in [22] is less accurate than the proposed one, since it disregards the second-order derivatives. Note that the expression provided in [22] to compute the covariance matrix results from (37) assuming that the objective function $J(\boldsymbol{x}, \boldsymbol{z})$ is multi-linear (i.e., $\boldsymbol{J}_{xx} \approx \boldsymbol{H}^T \boldsymbol{C}_z^{-1} \boldsymbol{H}$), which constitutes an approximation.

5.3. Unconstrained LAV Case

If the objective function in problem (1)–(3) corresponds to the LAV formulation, then:

$$J(\boldsymbol{y}) = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} |z_i - h_i(\boldsymbol{x})| . \tag{38}$$

If neither $\boldsymbol{l}(\boldsymbol{x}, \boldsymbol{z})$ nor $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$ constraints are taken into account, problem (1)–(3) can be written in an equivalent form by eliminating the absolute value function as:

$$J(\boldsymbol{y}) = \sum_{i=1}^{m} y_i \tag{39}$$

subject to:

$$z_i - h_i(\boldsymbol{x}) \leq y_i, \; i = 1, \ldots, m \tag{40}$$

$$-z_i + h_i(\boldsymbol{x}) \leq y_i, \; i = 1, \ldots, m . \tag{41}$$

For the sake of clarity, the weighting matrix $\boldsymbol{W}$ is considered to be the identity matrix in the derivations below.

The solution of problem (39)–(41) is given by n measurements which fit perfectly the state estimate, hereinafter called *basic* measurements. These measurements have zero residuals, whereas the remaining $m-n$ measurements can exhibit nonzero residuals (*non-basic* measurements). Let Γ^{C} and Γ^{S} be the sets of active and inactive constraints, respectively; i.e., set Γ^{C} (Γ^{S}) comprises the basic (non-basic) measurements [30].

Matrices in (15) become:

$$\boldsymbol{J}_{xx} = \sum_{\forall k \in \Gamma^{\mathrm{C}}} \lambda_k^* \nabla_{\boldsymbol{xx}} f_k(\hat{\boldsymbol{x}}, \boldsymbol{z}) \tag{42}$$

$$\boldsymbol{J}_{xz} = \emptyset \tag{43}$$

$$\boldsymbol{F}_x = \boldsymbol{H}_{\mathrm{C}} \tag{44}$$

$$\boldsymbol{F}_z = -\boldsymbol{I}^* \tag{45}$$

where $\boldsymbol{H}_{\mathrm{C}}$ is the $n \times n$ Jacobian measurement matrix corresponding to the basic measurements (null residuals), and $\boldsymbol{I}^*$ is a $n \times m$ matrix whose columns corresponding to non-basic measurements are zero, and the columns corresponding to basic measurements constitute a $n \times n$ identity matrix.

It can be shown that (see A):

$$\boldsymbol{M}_{xz} = \boldsymbol{H}_{\mathrm{C}}^{-1} \boldsymbol{I}^* \ . \tag{46}$$

The rows of matrix $\boldsymbol{H}$ can be sorted by sets Γ^{C} and Γ^{S}:

$$\boldsymbol{H} = \left[\begin{array}{c} \boldsymbol{H}_{\mathrm{C}} \\ \hline \boldsymbol{H}_{\mathrm{S}} \end{array} \right] \tag{47}$$

and, analogously, the columns of matrix $\boldsymbol{M}_{xz}$ can also be sorted:

$$\boldsymbol{M}_{xz} = \left[\boldsymbol{H}_{\mathrm{C}}^{-1} \mid \boldsymbol{0} \right] \ . \tag{48}$$

Using (6) and (11), matrix $\boldsymbol{\Omega}$ is obtained:

$$\begin{aligned} \boldsymbol{S} &= \boldsymbol{I} - \boldsymbol{H}\boldsymbol{M}_{xz} \\ &= \boldsymbol{I} - \begin{bmatrix} \boldsymbol{H}_{\mathrm{C}} \\ \hline \boldsymbol{H}_{\mathrm{S}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{H}_{\mathrm{C}}^{-1} \mid \boldsymbol{0} \end{bmatrix} \\ &= \boldsymbol{I} - \begin{bmatrix} \boldsymbol{I} \quad \mid \boldsymbol{0} \\ \hline \boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1} \mid \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \quad \mid \boldsymbol{0} \\ \hline -\boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1} \mid \boldsymbol{I} \end{bmatrix} \end{aligned} \quad (49)$$

$$\begin{aligned} \boldsymbol{\Omega} &= \boldsymbol{S}\boldsymbol{C}_z\boldsymbol{S}^T \\ &= \begin{bmatrix} \boldsymbol{0} \quad \mid \boldsymbol{0} \\ \hline -\boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1} \mid \boldsymbol{I} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{0} \mid \left(-\boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1}\right)^T \\ \hline \boldsymbol{0} \mid \quad \boldsymbol{I} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{0} \mid \ \boldsymbol{0} \\ \hline \boldsymbol{0} \mid \boldsymbol{\Omega}_{\mathrm{C}} \end{bmatrix} \end{aligned} \quad (50)$$

where

$$\boldsymbol{\Omega}_{\mathrm{C}} = \boldsymbol{I} - \boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1}(-\boldsymbol{H}_{\mathrm{S}}\boldsymbol{H}_{\mathrm{C}}^{-1})^T \ . \quad (51)$$

5.3.1. Comparison with the Approach in [16]

In this case, the final expression obtained using the proposed method corresponds to the equation traditionally used in the technical literature, as reported in [15].

In case of considering constraints, the proposed sensitivity-based method can also be applied. However, to the best of our knowledge, there is no method reported in the technical literature to compute matrix $\boldsymbol{\Omega}$ for the LAV estimator including constraints.

6. Case Study

In this section, we study the computational performance and numerical accuracy of the proposed method and other approaches, by means of several case studies. In each case, a different estimator is applied to the IEEE 30-bus system [31], and the residual covariance matrix is computed using the proposed approach and others. Results are then compared.

In order to obtain statistically sound conclusions, one hundred measurement scenarios are considered for each case. Measurement scenarios are generated adding zero-mean Gaussian errors to true measurements computed using a power flow solution. Considered measurements comprise voltages, and active and reactive power flows, providing a measurement redundancy ratio of $r = 112/(39 \cdot 2 - 1) = 1.90$.

6.1. Matrix $\mathbf{\Omega}$ *Computation Approaches*

Three cases are studied, each one considering a different estimator: WLS, LAV, and QCC. The number of available methods to compute the residual covariance matrix differs depending on the estimator employed. The two following approaches can be applied for these three estimators:

1. *Sensitivity Approach* (SA): This method is the one proposed in the paper, and relies on equation (11). This approach is usually more accurate than others reported in the literature, while requiring a similar computational effort.
2. *Numerical Approach* (NA): This method is based on the numerical calculation of the sensitivities using the formula:

$$f'(x) \approx \frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon} \tag{52}$$

   where ϵ is a small positive constant, and function $f'(\cdot)$ represents a state variable derivative with respect to a given measurement. Note that this method is computationally very expensive because it requires solving $2m$ state estimation problems, perturbing every measurement by $\pm\epsilon$, but provides accurate results if ϵ is small enough. This method is considered "exact" and a benchmark to compare the other methods. We use $\epsilon = 5 \times 10^{-5}$.

The proposed and others methods are compared with the numerical one (considered "exact"). Discrepancies between matrices are represented by means of the maximum and average relative errors.

Taking into account that a negligible $[\mathbf{\Omega}]_{(i,j)}$ coefficient can produce a very large relative error (and, therefore, can distort the results), we only compare coefficients ranging from $\pm 1\%$ to $\pm 100\%$ of the maximum absolute value coefficient, i.e., the relative error of any $[\mathbf{\Omega}]_{(i,j)}$ coefficient is considered if:

$$0.01 \max\left(|\mathbf{\Omega}|\right) \leq \left|[\mathbf{\Omega}]_{(i,j)}\right| \leq \max\left(|\mathbf{\Omega}|\right) \ . \tag{53}$$

Considering that the most important elements in matrix $\mathbf{\Omega}$ are the diagonal terms (used for the residual normalization process in bad data detection algorithms), some statistics are provided for these elements.

Additionally, the effect of considering or disregarding constraints $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z})$ are studied in each case. For the sake of simplicity, no inequality constraints are active in any case.

6.2. WLS Case

In this section, the WLS state estimator is considered. The technical literature provides two approaches to compute the residual covariance matrix:

1. *Traditional WLS* (TW): This method does not consider equality/inequality constraints, and it is based on equation (33) proposed in [21].
2. *Modified WLS* (MW): This method is a modification of the traditional WLS to include the influence of equality and active inequality constraints, [22], and it is based on equations (10) and (37).

Table 1 provides the average and maximum relative errors (in percentage) considering and not considering zero-injection buses for the three methods analyzed: sensitivity, traditional WLS, and modified WLS (SA, TW, and MW, respectively). Likewise, Table 2 provides the computation time statistics (minimum, average, maximum, and standard deviation) in performing the inversion process of the computation of matrix $\boldsymbol{M}_{xz}$.

[Table 1 about here.]

[Table 2 about here.]

The following observations are in order:

1. The average and maximum relative errors (for diagonal and non-diagonal terms) for SA and MW methods are negligible. Note that all these parameters are below 0.01% in all cases.
2. On the other hand, TW approach provides the same accuracy level than the MW algorithm if zero-injection buses are disregarded. If they are considered, the relative error increases significantly.
3. The numerical accuracy level provided by the SA method is slightly superior than that of the MW approach. Note that this minor improvement results from the consideration of the second-order derivatives in the calculation of matrix $\boldsymbol{M}_{xz}$.
4. From Table 2, note that the computational performance of three methods are very similar. If zero-injection buses are considered, the TW method is computationally lighter than the SA and MW ones, because it disregards equality constraints.

6.3. LAV Case

In this section, we study the LAV state estimator. The available method in the technical literature to compute the residual covariance matrix is:

1. *Traditional Approach* (TA): This method, reported in [15], is based on equation (51). However, this approach cannot be applied if equality constraints are considered.

[Table 3 about here.]

Table 3 provides the average and maximum relative errors (measured in percentage) for the proposed and traditional approaches (SA and TA, respectively).

The following observations are in order:

1. The relative errors (for diagonal and non-diagonal terms) for both approaches are negligible. All these values are below 0.0001% in all cases.
2. The numerical accuracy provided by both methods are the same if zero-injection buses are disregarded. This is due to the fact that the second-order derivatives do not affect the computation of matrix $\boldsymbol{M}_{xz}$ for the LAV estimator.
3. The only method capable of considering equality constraints is the proposed one, being highly accurate.

6.4. QCC Case

In this section, the quadratic-constant estimator is considered. For this estimator, to the best of our knowledge no method to compute the residual covariance matrix is available in the technical literature. However, the method proposed in this paper can be used. Thus, the only possible comparison is between the proposed method and the numerical approach.

The formulation of the QCC estimator is as follows:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad J(\boldsymbol{x}) = \sum_{i=1}^{m} s_i \tag{54}$$

subject to:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0} \tag{55}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{56}$$

$$s_i = \left\{ \begin{array}{ll} (h_i(\boldsymbol{x}) - z_i)^2 & \text{if } |h_i(\boldsymbol{x}) - z_i| \leq T \\ & \\ T^2 & \text{if } |h_i(\boldsymbol{x}) - z_i| \geq T \end{array} \right\} \forall i \tag{57}$$

where T is a parameter.

The aforementioned formulation can be transformed into a mixed integer nonlinear formulation, making use of binary variables to model equations (57). The precise details of the transformation are outside the scope of this paper.

[Figure 1 about here.]

Fig. 1 depicts the three objective functions considered in this section, and data from a particular measurement scenario of the QCC case. Note that for this QCC scenario data the majority of residuals are in the quadratic part of the curve, while some others are in the constant part. In this case, the parameter T is set to 0.025 pu.

Table 4 provides the average and maximum relative errors (measured in percentage) for the proposed approach (SA).

[Table 4 about here.]

The following observations are in order:

1. The average and maximum relative errors for both diagonal and non-diagonal terms are larger than the corresponding errors in the previous cases. This is due to the fact that the objective function derivatives are not continuous.
2. The average errors (for diagonal and non-diagonal terms) are low, being always inferior to 0.3%. However, the relative error of some non-diagonal terms is around 16%.
3. To obtain normalized residuals, only diagonal terms are needed. Note that the accuracy of diagonal terms is sufficiently high. The average relative error for diagonal terms is lower than 0.06%.

7. Conclusion

This paper proposes a novel technique to compute the residual covariance matrix and normalized residuals for a state estimator based on a mathematical programming formulation (e.g., WLS or LAV), considering constraints. This technique relies on the sensitivities of the state variables with respect to the measurements at the optimal solution of the estimation problem. The sensitivities are calculated through the solution of a linear system of equations, which results from the perturbation of the optimality conditions of the estimation problem.

Detailed numerical simulations show that the proposed method is an efficient and accurate technique to estimate the residual covariance matrix regardless of the considered estimator. The proposed technique that is estimator-independent is superior or similar to the estimator-dependent techniques proposed in the technical literature.

A. Computation of Matrix $\boldsymbol{M_{xz}}$ for the LAV Estimator

If the LAV estimator is used (see Section 5.3), the following expression holds true:

$$\boldsymbol{M}_{xz} = \boldsymbol{H}_{\mathrm{C}}^{-1}\boldsymbol{I}^{*} \tag{58}$$

as it is shown below.

Using (21) and (42), matrix $\boldsymbol{U}$ is computed as

$$\boldsymbol{U} = \left[\begin{array}{c|c} \boldsymbol{J}_{xx} & \boldsymbol{H}_{\mathrm{C}}^{T} \\ \hline \boldsymbol{H}_{\mathrm{C}} & \boldsymbol{0} \end{array}\right] . \tag{59}$$

Since matrix $\boldsymbol{U}$ is a square invertible matrix, then,

$$\begin{aligned} \boldsymbol{U}\cdot\boldsymbol{U}^{-1} &= \left[\begin{array}{c|c} \boldsymbol{J}_{xx} & \boldsymbol{H}_{\mathrm{C}}^{T} \\ \hline \boldsymbol{H}_{\mathrm{C}} & \boldsymbol{0} \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{E}_1 & \boldsymbol{E}_2 \\ \hline \boldsymbol{E}_3 & \boldsymbol{E}_4 \end{array}\right] \\ &= \left[\begin{array}{c|c} \boldsymbol{J}_{xx}\boldsymbol{E}_1 + \boldsymbol{H}_{\mathrm{C}}^{T}\boldsymbol{E}_3 & \boldsymbol{J}_{xx}\boldsymbol{E}_2 + \boldsymbol{H}_{\mathrm{C}}^{T}\boldsymbol{E}_4 \\ \hline \boldsymbol{H}_{\mathrm{C}}\boldsymbol{E}_1 & \boldsymbol{H}_{\mathrm{C}}\boldsymbol{E}_2 \end{array}\right] \\ &= \left[\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I} \end{array}\right] . \end{aligned} \tag{60}$$

On the other hand, matrix $\boldsymbol{H}_{\mathrm{C}}$ is also a square invertible matrix. Note that from (60) it can be deduced that:

$$\boldsymbol{H}_{\mathrm{C}}\boldsymbol{E}_2 = \boldsymbol{I} \iff \boldsymbol{E}_2 = \boldsymbol{H}_{\mathrm{C}}^{-1}. \tag{61}$$

Finally, using (23) and (61),

$$\begin{bmatrix} \boldsymbol{M}_{xz} \\ \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{z}} \end{bmatrix} = \left[\begin{array}{c} \boldsymbol{E}_1 \mid \boldsymbol{H}_{\mathrm{C}}^{-1} \\ \hline \boldsymbol{E}_3 \mid \boldsymbol{E}_4 \end{array}\right] \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}^* \end{bmatrix} \tag{62}$$

and, therefore, $\boldsymbol{M}_{xz} = \boldsymbol{H}_{\mathrm{C}}^{-1}\boldsymbol{I}^*$.

For the computation of matrix $\boldsymbol{M}_{xz}$, using equation (58), neither matrix $\boldsymbol{F}_{xz}$ nor $\boldsymbol{F}_{xx}$ is needed. This implies that second-order derivatives (i.e., nonlinearities in the objective function $J(\boldsymbol{x}, \boldsymbol{z})$) do not affect the computation of the residual covariance matrix.

B. Nomenclature

B.1. Parameters and Constants:

n Number of state variables.

m Number of measurements.

r Measurement redundancy ratio.

p Number of equality constraints.

q Number of inequality constraints.

q_Γ Number of binding inequality constraints.

ϵ Small positive constant.

T Adjustment parameter for the QCC estimator.

$\boldsymbol{z}$ Measurement vector.

$\boldsymbol{C}_z$ Measurement covariance matrix.

$\boldsymbol{W}$ Weighting matrix.

$\boldsymbol{I}$ Identity matrix.

B.2. Functions and Functional Matrices:

$J(\cdot)$ Objective function.

$\boldsymbol{l}(\cdot)$ Nonlinear equality constraint vector.

$\boldsymbol{g}(\cdot)$ Nonlinear inequality constraint vector.

$\boldsymbol{f}(\cdot)$ Equality and binding inequality constraint vector.

$\boldsymbol{h}(\cdot)$ Nonlinear functional vector.

$\boldsymbol{H}$ Jacobian measurement matrix.

$\boldsymbol{H}_{\mathrm{C}}$ Jacobian matrix of basic measurements.

$\boldsymbol{H}_{\mathrm{S}}$ Jacobian matrix of nonbasic measurements.

$\boldsymbol{F}_{\mathrm{x}}$ Jacobian of $\boldsymbol{f}(\cdot)$ with respect to state variables.

$\boldsymbol{F}_{\mathrm{z}}$ Jacobian of $\boldsymbol{f}(\cdot)$ with respect to measurements.

$\boldsymbol{M}_{xz}$ Matrix of derivatives of state variables with respect to measurements.

$\boldsymbol{S}$ Sensitivity matrix.

$\boldsymbol{\Omega}$ Residual covariance matrix.

B.3. State, Residual and Dual Vectors:

$\boldsymbol{x}$ State vector.

$\boldsymbol{x}^{\mathrm{true}}$ True state vector.

$\hat{\boldsymbol{x}}$ Estimated state vector.

$\boldsymbol{r}$ Residual vector.

$\boldsymbol{r}^N$ Normalized residual vector.

$\boldsymbol{\lambda}$ Dual variable vector related to $\boldsymbol{f}(\cdot)$.

B.4. Sets:

Γ^{C} Set of basic measurements.

Γ^{S} Set of nonbasic measurements.

B.5. Estimators:

WLS Weighted Least Squares.

LAV Least Absolute Value.

LMS Least Median of Squares.

LTS Least Trimmed of Squares.

QCC Quadratic-Constant Criterion.

QLC Quadratic-Linear Criterion.

Acknowledgements

E. Caro, A.J. Conejo, and R. Mínguez are partly supported by the Junta de Comunidades de Castilla-La Mancha, through project PCI08-0102; and by the Ministry of Science and Education of Spain through CICYT Projects DPI2006-08001. R. Mínguez is partly supported by the Spanish Ministry of Science and Innovation (MCI) through the "Ramon y Cajal" program.

References

[1] F. C. Schweppe, J. Wildes, Power system static state estimation. Part I: Exact model, IEEE Trans. Power App. Syst. 89 (1)(1970) 120–125.

[2] F. C. Schweppe, D. Rom, Power system static state estimation. Part II: Approximate model, IEEE Trans. Power App. Syst. 89 (1)(1970) 125–130.

[3] F. C. Schweppe, Power system static state estimation. Part III: Implementation, IEEE Trans. Power App. Syst. 89 (1)(1970) 130–135.

[4] F. C. Schweppe, E. J. Handschin, Static state estimation in electric power systems, Proc. IEEE 62 (7)(1974) 972–982.

[5] R. Larson, W. Tinney, L. Hadju, D. Piercy, State estimation in power systems. Part II: Implementations and applications, IEEE Trans. Power App. Syst. 89 (3)(1970) 353–362.

[6] A. García, A. Monticelli, P. Abreu, Fast decoupled state estimation and bad data processing, IEEE Trans. Power App. Syst. 98 (9-10)(1979) 1645–1652.

[7] A. Monticelli, A. García, Fast decoupled state estimators, IEEE Trans. Power Syst. 5 (5)(1990) 556–564.

[8] J. J. Allemong, L. Radu, A. M. Sasson, A fast and reliable state estimation algorithm for AEP's new control center, IEEE Trans. Power App. Syst. 101 (4)(1982) 933–944.

[9] L. Holten, A. Gjelsvik, S. Aam, F. F. Wu, W. H. E. Liu, Comparison of different methods for state estimation, IEEE Trans. Power Syst. 3 (11)(1988) 1798–1806.

[10] A. Monticelli, Electric power system state estimation, Proc. IEEE 88 (2000) 262–282.

[11] A. Abur, A. G. Expósito, Electric Power System State Estimation. Theory and Implementations, Marcel Dekker, 2004.

[12] E. Caro, A. J. Conejo, R. Mínguez, Optimization Advances in Electric Power Systems, Nova Science Publishers, Inc, pp. 1–26.

[13] R. Baldick, K. A. Clements, Z. Pinjo-Dzigal, P. W. Davis, Implementing nonquadratic objective functions for state estimation and bad data rejection, IEEE Trans. Power Syst. 12 (2)(1997) 376–382.

[14] F. Zhuang, R. Balasubramanian, Bad data suppression in power system state estimation with a variable quadratic-constant criterion, IEEE Trans. Power App. Syst. PAS-104 (5)(1985) 857–863.

[15] A. Abur, A bad data identification method for linear programming state estimation, IEEE Trans. Power Syst. 5 (8)(1990) 894–901.

[16] M. K. Çelic, A. Abur, A robust WLAV state estimator using transformations, IEEE Trans. Power Syst. 7 (2)(1992) 106–113.

[17] A. Abur, Least absolute value state estimation with equality and inequality constraints, IEEE Trans. Power Syst. 8 (5)(1993) 680–686.

[18] L. Mili, M. Cheniae, N. Vichare, , P. Rousseeuw, Algorithms for least median of squares state estimation of power systems, in: Proc. 35th IEEE Midwest Symposium on Circuits and Syst., Washington, D.C., pp. 1276–1283.

[19] L. Mili, V. Phaniraj, P. Rousseeuw, Least median of squares estimation in power systems, IEEE Trans. Power Syst. 6 (5)(1991) 511–523.

[20] L. Mili, M. Cheniae, P. Rousseeuw, Robust state estimation of electric power systems, IEEE trans. circuits and systems 41 (5)(1994) 348–358.

[21] A. Monticelli, A. Garcia, Reliable bad data processing for real-time state estimation, IEEE Trans. Power App. Syst. PAS-102 (5)(1983) 1126–1137.

[22] F. F. Wu, W. H. E. Liu, S. Lun, Observability analysis and bad data processing for state estimation with equality constraints, IEEE Trans. Power Syst. 3 (5)(1988) 541–548.

[23] P. J. Huber, Robust Statistics, New York. John Wiley, 1981.

[24] R. A. Maronna, R. D. Martin, V. J. Yohai, Robust Statistics: Theory and Methods, John Wiley, 2006.

[25] R. Mínguez, A. J. Conejo, State estimation sensitivity analysis, IEEE Trans. Power Syst. 22 (8)(2007) 1080–1091.

[26] E. Castillo, A. J. Conejo, C. Castillo, R. Mínguez, D. Ortigosa, A perturbation approach to sensitivity analysis in nonlinear programming, Journal of Optimization Theory and Applications 128 (1)(2006) 49–74.

[27] A. Conejo, E. Castillo, R. Mínguez, R. García-Bertrand, Decomposition techniques in mathematical programming. Engineering and science applications, New York. Springer Berlin Heidelberg, 2006.

[28] E. Caro, A. Conejo, R. Mínguez, Power system state estimation considering measurement dependencies, IEEE Trans. Power Syst. 24 (11)(2009) 1875–1885.

[29] E. Caro, A. Conejo, J. Morales, R. Mínguez, Calculation of measurement correlations using point estimate, IEEE Trans. Power Del. (2010) Accepted for publication.

[30] E. Castillo, C. Castillo, A. S. Hadi, R. Mínguez, Duality and local sensitivity analysis in least squares, minimax, and least absolute values regressions, Journal of Statistical Computation and Simulation 78 (2008) 887–909.

[31] Power systems test case archive. [Online]. Available on: *http://www.ee.washington.edu/research/pstca/.*

List of Figures

Table 1: Relative error (%) for the WLS case study.

	Disregarding zero-injec. buses				Considering zero-injec. buses			
	All terms		Diagonal terms		All terms		Diagonal terms	
	Avg.	Max.	Avg.	Max.	Avg.	Max.	Avg.	Max.
SA	<0.001	0.002	<0.0001	<0.001	<0.001	0.007	<0.0001	0.001
TW	<0.001	0.008	0.0004	0.007	39	2000	20	100
MW	<0.001	0.008	0.0004	0.007	<0.001	0.004	0.0002	0.002

Table 2: Computation time (in milliseconds) for the WLS case study.

	Disregarding zero-injec. buses				Considering zero-injec. buses			
	Minim.	Averg.	Maxim.	Std.	Minim.	Averg.	Maxim.	Std.
SA	0.91	1.07	1.23	0.22	1.33	1.35	1.66	0.03
TW	0.92	0.93	0.93	0.01	0.88	0.90	0.92	0.01
MW	0.93	0.95	0.96	0.02	1.30	1.32	1.35	0.01

Table 3: Relative error (%) for the LAV case study.

	Disregarding zero-injec. buses				Considering zero-injec. buses			
	All terms		Diagonal terms		All terms		Diagonal terms	
	Avg.	Max.	Avg.	Max.	Avg.	Max.	Avg.	Max.
SA	$5{\times}10^{-7}$	$2{\times}10^{-5}$	$6{\times}10^{-7}$	$4{\times}10^{-6}$	$3{\times}10^{-7}$	$5{\times}10^{-5}$	$3{\times}10^{-6}$	$5{\times}10^{-5}$
TA	$5{\times}10^{-7}$	$2{\times}10^{-5}$	$6{\times}10^{-7}$	$4{\times}10^{-6}$	–	–	–	–

Table 4: Relative error (%) for the QCC case study.

	Disregarding zero-injec. buses				Considering zero-injec. buses			
	All terms		Diagonal terms		All terms		Diagonal terms	
	Avg.	Max.	Avg.	Max.	Avg.	Max.	Avg.	Max.
SA	0.04	4.1	<0.01	0.07	0.28	16.1	0.06	1.5

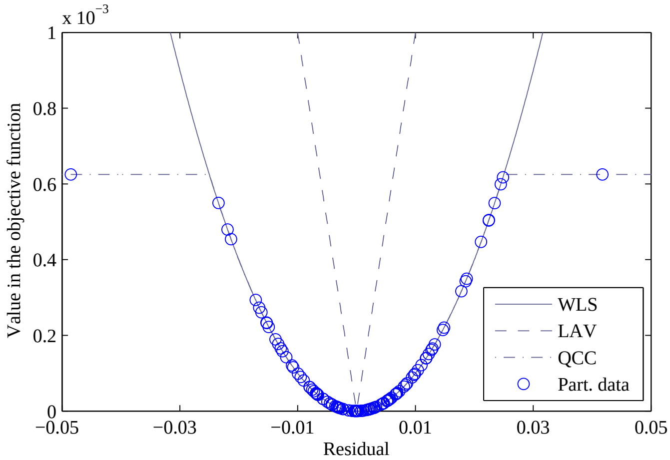

Figure 1: Objective functions of several estimators.